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The dressing factor and crossing equations

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Abstract
We utilize the DHM integral representation for the BES dressing factor of the world-sheet S-matrix of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ light-cone string theory, and the crossing equations to fix the principal branch of the dressing factor on the rapidity torus. The results obtained are further used, in conjunction with the fusion procedure, to determine the bound state dressing factor of the mirror theory. We convincingly demonstrate that the mirror bound state S-matrix found in this way does not depend on the internal structure of a bound state solution employed in the fusion procedure. This welcome feature is in perfect parallel to string theory, where the corresponding bound state S-matrix has no bearing on bound state constituent particles as well. The mirror bound state S-matrix we found provides the final missing piece in setting up the TBA equations for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror theory.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The exact finite-size spectrum of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring remains one of the most important challenges in the AdS/CFT correspondence [1]. In the light-cone gauge the string sigma model is formulated on a two-dimensional cylinder of finite circumference proportional to the light-cone momentum. When the light-cone momentum tends to infinity, the string world-sheet decompactifies and, under the assumption of quantum integrability of the light-cone model, one can apply factorized scattering theory [2] and the Bethe ansatz to capture the spectrum. A lot of remarkable progress has been achieved in this way; for the recent reviews and extensive list of references, see, e.g., [3–5].

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On the infinite world-sheet, the string sigma model exhibits massive excitations which transform in the tensor product of two fundamental representations of the $\mathfrak{psu}(2|2)$ superalgebra enhanced by two central charges, both dependent on the generator of the world-sheet momentum [6, 7]. The symmetry algebra severely constrains the matrix form of the two-particle S-matrix [6] and, together with the Yang–Baxter equation and the requirement of generalized physical unitarity [8, 9], leads to its unique determination up to an overall scalar function of particle momenta. This function includes, as its most non-trivial piece, the so-called dressing factor $\sigma = \exp(\mathrm{i}\theta)$, where θ is the dressing phase [10]. Taking into account that the light-cone S-matrix is compatible with the assumption of crossing symmetry, one finds that the latter implies certain functional relations for the dressing factor, known as the crossing equations [11]. A solution to these equations has been found in the strong coupling expansion in terms of an asymptotic series [12]. The corresponding dressing factor appears to be compatible with all available classical and perturbative string data [10], [13–19] and it passed a number of further non-trivial checks, see e.g. [20, 21].

The dressing factor suitable for the weak coupling expansion has also been proposed by assuming a certain analytic continuation of its strong-coupling counterpart [22], and in what follows we refer to it as the Beisert–Eden–Staudacher (BES) dressing factor. The proposal was successfully confronted against direct field-theoretic perturbative calculations [23–25]. However, it has not been shown so far that the BES dressing factor satisfies crossing equations for finite values of the string coupling. Filling this gap is among the goals of this paper.

Another important issue which motivates our present effort is related to the finite-size spectral problem. When the light-cone momentum is finite, the description of the string spectrum in terms of the Bethe equations [26, 27] fails, and one has to resort to new methods to find a solution of the corresponding spectral problem. In this respect an exciting possibility is offered by the thermodynamic Bethe ansatz approach (TBA) originally developed for relativistic integrable models [28]. In essence, the TBA allows one to relate the finite-size ground state energy in a two-dimensional integrable model with the free energy (or, depending on the boundary conditions, with Witten's index) in the so-called mirror model that is obtained from the original model by a double Wick rotation. A peculiarity of the non-relativistic case, as provided by the string sigma model, is that the original and the mirror Hamiltonians are not the same. This brings certain subtleties in setting up the corresponding TBA approach [9].

Recently, we have derived a set of the TBA equations for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model [29, 30]. A parallel development took place in the works [31, 32] and [33], where also the so-called Y-system has been proposed. The Y-system represents a set of local equations which is believed to contain the whole spectrum (the ground and excited states), and it is obtained from the original infinite set of TBA equations by taking a certain projection. In one respect, the undertaken derivations of the mirror TBA equations and the associated Y-system remain incomplete. To completely define the TBA equations and, therefore, to make them a working device, one has to understand the so far unknown analytic properties of the dressing factor in the kinematic region of the mirror theory.

To be precise, the BES dressing phase is represented by a double series convergent in the region $|x_1^\pm| > 1$ and $|x_2^\pm| > 1$, where $x_{1,2}^\pm$ are kinematic parameters related to the first and the second particle, respectively. This series admits an integral representation found by Dorey, Hofman and Maldacena (DHM) [34] which is valid in the same region of kinematic parameters and for finite values of the string coupling.[4] To find the BES dressing phase in other kinematic regions of interest (in the mirror region), one has to analytically continue the

[4] Another useful integral representation was found in [35].

DHM integral representation beyond $\left|x^{\pm}_{1,2}\right| > 1$. Understanding this continuation is precisely the subject of this paper.

Regarding the dressing phase as a multi-valued function, we want to fix its particular analytic branch which satisfies the crossing equations. In fact, continuation compatible with crossing is the main rationale behind our treatment. Consideration of crossing requires us to associate with each of the particles (bound states) the so-called rapidity torus (the z-torus) which uniformizes the corresponding dispersion relation [11]. The analytically continued dressing phase should be then understood as a function $\theta(z_1, z_2)$, where z_1 and z_2 belong to the corresponding tori; see section 2 for a more precise definition. This function must satisfy the crossing equation in each of its arguments for z_1, z_2 being anywhere on the tori. Thus, to construct $\theta(z_1, z_2)$, we choose an analytic continuation path on the z-torus which starts in the particle region, where $|x^{\pm}(z)| > 1$, and penetrates the other regions of the z-torus. Following this path, we properly account for the change of the DHM integral representation to guarantee the continuity of our resulting function. In this way, we build up the analytic continuation of the dressing phase to all possible kinematic regions and then verify the fulfillment of the crossing equations.

We further study the analytic continuation of the dressing factor for bound states of the string model. This dressing factor is obtained by fusing the dressing factors of bound state constituent particles, all of them being in the region $|x^{\pm}(z)| > 1$ of the elementary z-torus. The dressing factor is then analytically continued into the whole bound state z-torus in a way compatible with the corresponding crossing equations.

In view of applications to the TBA program, we also determine the dressing factor for bound states of the mirror model by fusing the dressing factors of constituent mirror particles. Since some of these particles fall necessarily outside the region $|x^{\pm}(z)| > 1$, in the fusion procedure we use their analytically continued expressions. Equations defining a Q-particle bound state admit 2^{Q-1} different solutions and, since any of these solutions can be used in the fusion procedure, this raises a question about uniqueness of the final result. To find an answer, it is important to realize that dealing with mirror TBA equations, we are mainly interested not in the bound state dressing factor $\sigma^{QQ'}$ itself but rather in the following quantity:

$$\Sigma^{QQ'} = \sigma^{QQ'} \prod_{j=1}^{Q}\prod_{k=1}^{Q'} \frac{1 - \frac{1}{x_j^+ z_k^-}}{1 - \frac{1}{x_j^- z_k^+}},$$

which logarithmic derivative appears as one of the TBA kernels [30]. Here $x_j^{\pm}$ and $z_k^{\pm}$ are the kinematical parameters of the constituent particles corresponding to Q- and Q'-particle bound states, respectively. The *improved* factor $\Sigma^{QQ'}$ originates from the fusion of the scalar factors of mirror theory scattering matrices corresponding to bound state constituent particles. By evaluating $\Sigma^{QQ'}$ on a particular bound state solution, we then show that the resulting expression depends on the bound state kinematic parameters only and that all the dependence on constituent particles completely disappears. This indicates that $\Sigma^{QQ'}$ is the same for all bound state solutions, as we also confirm by explicitly evaluating it on yet another solution. In this respect, $\Sigma^{QQ'}$ appears as good as the corresponding bound state dressing factor of the original string theory. We derive an explicit formula (6.14) for $\Sigma^{QQ'}$ which can be used to complete the mirror TBA equations.

The paper is organized as follows. In the next section, we recall the necessary facts about the dressing phase, rapidity torus and the crossing equations for both fundamental particles and the bound states. In section 3, we introduce our basic building blocks—the Φ and Ψ-functions—and study their analytic properties. In section 4, we determine a particular analytic continuation of the dressing phase for fundamental particles and verify that it obeys the crossing equations. In section 5, we do the same for bound states of string theory. In principle, the material of section 5 contains the one of section 4, as fundamental particles can be regarded as one-particle bound states. However, the discussion in section 4 gives a clear picture of how the analytic continuation is constructed and, therefore, provides a good preparation for understanding the technically more involved issue of the bound states. In section 6, we determine the bound state dressing factor of the mirror theory and argue that it has the same universal form regardless of a bound state solution chosen for its construction. Finally, we conclude by discussing the results obtained. In four appendices, we collected some identities satisfied by Ψ-functions, as well as technical details of the derivations presented in the main body of the paper.

2. The dressing phase and crossing equations

In the light-cone gauge, the string sigma model exhibits a massive spectrum. The corresponding particles transform in the tensor product of two fundamental multiplets of the centrally extended $\mathfrak{su}(2|2)$ superalgebra. Besides the fundamental particles, the asymptotic spectrum also contains their bound states which manifest themselves as poles of the scattering matrix for fundamental particles. A Q-particle bound state [36] transforms in the tensor product of two $4Q$-dimensional atypical totally symmetric multiplets of the centrally extended $\mathfrak{su}(2|2)$ superalgebra [37, 38]. Both the fundamental and the bound state S-matrices are determined by kinematical symmetries and additional physicality requirements up to an overall scalar factor. Upon normalizing the kinematically determined S-matrix in a certain (canonical) way, the corresponding scalar factor acquires an absolute meaning and therefore becomes an important dynamical characteristic of the model. For the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring, the scalar factor is proportional to the dressing factor σ, the latter can be expanded over local conserved charges of the model [10]. By unitarity $\sigma = \exp(\mathrm{i}\theta)$, where θ is known as the *dressing phase*. This section has an introductory character; its purpose is to recall the relevant facts about the dressing phase as well as to introduce the necessary notation.

2.1. Uniformization tori

As was argued in [11], the world-sheet S-matrix appears to be compatible with an assumption of crossing symmetry that corresponds to replacing a particle for its anti-particle. Implementation of this symmetry leads to a non-trivial functional relation for the dressing phase.

Similar to the more familiar relativistic case, a rigorous treatment of crossing symmetry requires finding a Riemann surface which uniformizes the dispersion relation of the model. For fundamental particles this uniformization is achieved by introducing a torus with real and imaginary periods given by

$$2\omega_1 = 4\mathrm{K}(k), \quad 2\omega_2 = 4\mathrm{i}\mathrm{K}(1-k) - 4\mathrm{K}(k), \tag{2.1}$$

respectively [11, 39] . Here $\mathrm{K}(k)$ stands for the complete elliptic integral of the first kind. The elliptic modulus $k = -4g^2$, where g is the string tension related to the 't Hooft coupling λ as $g = \frac{\sqrt{\lambda}}{2\pi}$.

Analogously, the dispersion relation corresponding to a Q-particle bound state is also uniformized by an elliptic curve whose periods are obtained by replacing in equation (2.1) the coupling constant g with g/Q. Neglecting for the moment the dressing factor, the *canonically* normalized S-matrix that describes scattering of Q- and M-particle bound states turns out to be a meromorphic matrix function $S_{QM}(z_1, z_2)$ defined in the product of the corresponding

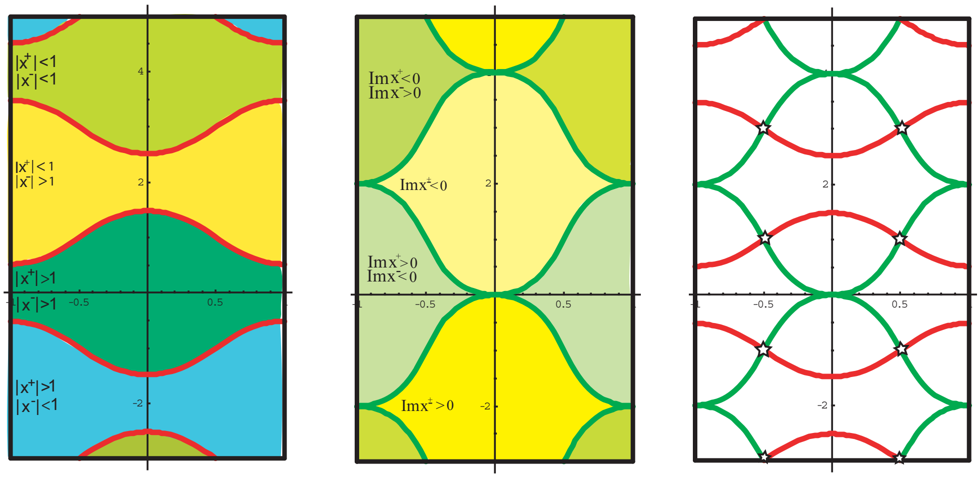


Figure 1. On the left figure the torus is divided by the curves $|x^+| = 1$ and $|x^-| = 1$ into four non-intersecting regions. The middle figure represents the torus divided by the curves $\mathrm{Im}(x^+) = 0$ and $\mathrm{Im}(x^-) = 0$, also in four regions. The right figure contains all the curves of interest.

tori [9, 40]. In what follows we will refer to a torus-supporting fundamental or bound state representations as the z-torus.

In addition, we will also need the variables $x^\pm$ which satisfy the following constraint:

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2\mathrm{i}}{g}, \tag{2.2}$$

and which are related to the torus variable z through $\frac{x^+}{x^-} = \exp(\mathrm{i}\,2\,\mathrm{am}\,z)$, where $\mathrm{am}\,z$ is the Jacobi amplitude. The functions $x^\pm(z)$ are meromorphic on the z-torus corresponding to fundamental particles. The torus itself can be divided into four non-intersecting regions in two different ways: either by curves $|x^\pm| = 1$ or by curves $\mathrm{Im}\,x^\pm = 0$, see figure 1. These divisions will play an important role in our subsequent discussion of the dressing phase.

Under the crossing symmetry transformation, the variables $x^\pm$ undergo the inversion: $x^\pm \to 1/x^\pm$. In terms of variable z, this map is realized as a shift by the imaginary half-period: $z \to z \pm \omega_2$ [11]. The dressing phase, however, is not periodic on the z-torus and should be considered as a function on the product of two z-planes. The crossing symmetry allows one to define the dressing phase on the product of two complex planes from its knowledge on the z-torus, the latter being regarded as the fundamental domain. As we will see later on, inside this fundamental domain, the dressing phase has an infinite number of cuts.

2.2. *Dressing and crossing for fundamental particles*

The dressing phase for fundamental particles is a function of the constrained variables $x^\pm$: $\theta \equiv \theta\left(x_1^+, x_1^-, x_2^+, x_2^-\right)$ and, in the region $\left|x_{1,2}^\pm\right| > 1$, it admits the following double series expansion [10, 41]:

$$\theta\left(x_1^+, x_1^-, x_2^+, x_2^-\right) = \sum_{r=2}^{\infty} \sum_{\substack{s>r \\ r+s=\text{odd}}}^{\infty} c_{r,s}(g)\left[q_r\left(x_1^\pm\right) q_s\left(x_2^\pm\right) - q_r\left(x_2^\pm\right) q_s\left(x_1^\pm\right)\right], \tag{2.3}$$

where the local conserved charges $q_r(x^\pm)$ are

$$q_r\left(x_k^-, x_k^+\right) = \frac{\mathrm{i}}{r-1}\left[\left(\frac{1}{x_k^+}\right)^{r-1} - \left(\frac{1}{x_k^-}\right)^{r-1}\right]. \tag{2.4}$$

Here the coefficients $c_{r,s}(g)$ are non-trivial real functions of the string tension admitting a well-defined asymptotic expansion for large g. The double series representation (2.3) implies that the dressing phase in this region can be written [42] via a single skew-symmetric function χ of two variables

$$\chi(x_1, x_2) = -\sum_{r=2}^{\infty} \sum_{\substack{s>r \\ r+s=\text{odd}}}^{\infty} \frac{c_{r,s}(g)}{(r-1)(s-1)} \left[\frac{1}{x_1^{r-1} x_2^{s-1}} - \frac{1}{x_2^{r-1} x_1^{s-1}} \right] \tag{2.5}$$

as a sum of four terms

$$\theta\left(x_1^+, x_1^-, x_2^+, x_2^-\right) = \chi\left(x_1^+, x_2^+\right) - \chi\left(x_1^+, x_2^-\right) - \chi\left(x_1^-, x_2^+\right) + \chi\left(x_1^-, x_2^-\right). \tag{2.6}$$

At any given order in the asymptotic $1/g$ expansion, the double series defining χ is convergent for $|x_{1,2}| > 1$. For the z-torus in figure 1, conditions $|x^{\pm}(z)| > 1$ single out a green region with the shape of a 'fish'. Thus, every term in the strong coupling asymptotic expansion of the dressing phase $\theta(z_1, z_2)$ is a well-defined function provided both z_1 and z_2 belong to the 'fish'. The set of points on the z-torus obeying $|x^{\pm}(z)| > 1$ will be called the particle region.

Assuming the functional form (2.3), the equations implied by crossing symmetry (to be discussed below) can be solved perturbatively in the strong coupling expansion, the coefficients $c_{r,s}(g)$ emerge in the form of an *asymptotic series* [12]. On the other hand, the coefficients $c_{r,s}(g)$ admit a *convergent* small g expansion

$$c_{r,s}(g) = g \sum_{n=r+s-3}^{\infty} g^n c_{r,s}^{(n)}.$$

The weak coupling coefficients $c_{r,s}^{(n)}$ have been determined from their strong coupling cousins by assuming a certain analytic continuation procedure [22].

By using the weak coupling expressions for $c_{r,s}(g)$ and the series (2.5), in the work [34] an integral representation for χ valid for finite values of g was obtained. It is given by the following double integral:

$$\chi(x_1, x_2) = \mathrm{i} \oint \frac{\mathrm{d}w_1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}w_2}{2\pi \mathrm{i}} \frac{1}{(w_1 - x_1)(w_2 - x_2)} \log \frac{\Gamma\left[1 + \frac{\mathrm{i}}{2} g\left(w_1 + \frac{1}{w_1} - w_2 - \frac{1}{w_2}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2} g\left(w_1 + \frac{1}{w_1} - w_2 - \frac{1}{w_2}\right)\right]}, \tag{2.7}$$

where integrations are performed over the unit circles. The integral representation (2.7) holds for $|x_{1,2}| > 1$ and, therefore, from the point of view of the z-torus, for any finite value of g, it renders the dressing phase (2.6) a well-defined function on the product of two-particle regions. We will call equation (2.7) the DHM integral representation.

Assuming that the dressing phase $\theta(z_1, z_2)$ is defined on the whole z-torus, the functional equations implied by the crossing symmetry read

$$\theta(z_1, z_2) + \theta(z_1 + \omega_2, z_2) = \frac{1}{\mathrm{i}} \log \left[\frac{x_2^-}{x_2^+} h(x_1, x_2) \right], \tag{2.8}$$

$$\theta(z_1, z_2) + \theta(z_1, z_2 - \omega_2) = \frac{1}{\mathrm{i}} \log \left[\frac{x_1^+}{x_1^-} h(x_1, x_2) \right], \tag{2.9}$$

where

$$h(x_1, x_2) = \frac{x_1^- - x_2^+}{x_1^- - x_2^-} \frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}. \tag{2.10}$$

Under the double crossing one gets, for instance,

$$\theta(z_1+2\omega_2,z_2)-\theta(z_1,z_2)=\frac{1}{\mathrm{i}}\log h_{\mathcal{D}}(x_1,x_2), \tag{2.11}$$

where

$$h_{\mathcal{D}}(x_1,x_2)=\frac{\left(x_1^+-x_2^+\right)\left(x_1^--x_2^-\right)}{\left(x_1^+-x_2^-\right)\left(x_1^--x_2^+\right)}\frac{\left(1-\frac{1}{x_1^-x_2^+}\right)\left(1-\frac{1}{x_1^+x_2^-}\right)}{\left(1-\frac{1}{x_1^+x_2^+}\right)\left(1-\frac{1}{x_1^-x_2^-}\right)}, \tag{2.12}$$

which shows that $2\omega_2$ is not a period of the dressing phase.

It should be stressed that so far crossing symmetry has been imposed on the dressing phase in the asymptotic sense only, and this led, through the analytic continuation procedure from strong to weak coupling, to the finite g representation defined with the help of (2.7). Verification of the crossing symmetry for finite g, i.e. not in the asymptotic sense, constitutes an open problem. Its solution relies on finding an analytic continuation of the dressing phase from the particle region to the whole z-torus which is compatible with the crossing symmetry.

Our final remark concerns the integral representation (2.7). Formula (2.7) holds for $|x_{1,2}|>1$ but, in fact, it can be used to continue the function χ for each of its arguments slightly inside the unit circle. Indeed, the integration contours can be chosen to be circles of any radius $r>r_{\rm cr}\equiv\sqrt{1+\frac{1}{4g^2}}-\frac{1}{2g}$, and, therefore, the integral representation above can be extended for $|x_{1,2}|>r_{\rm cr}$. Moreover, if one of the circles in (2.7) is of unit radius then the radius of the second circle can be reduced up to $\sqrt{1+\frac{1}{g^2}}-\frac{1}{g}$.

2.3. Dressing and crossing for bound states

The dressing phase which describes scattering of Q- and M-particle bound states can be obtained from the dressing phase for fundamental particles by means of the fusion procedure [43, 44]. The fused dressing phase θ^{QM} is given by the same formulas (2.6) and (2.7), where now the variables $x_1^{\pm}$ and $x_2^{\pm}$ are associated with the Q- and M-particle bound states, respectively,

$$x_1^++\frac{1}{x_1^+}-x_1^--\frac{1}{x_1^-}=\frac{2\mathrm{i}}{g}Q,\quad x_2^++\frac{1}{x_2^+}-x_2^--\frac{1}{x_2^-}=\frac{2\mathrm{i}}{g}M\,. \tag{2.13}$$

The crossing equations for the dressing factor describing scattering of Q- and M-particle bound states are [40][5]

$$\sigma^{QM}(z_1,z_2)\sigma^{QM}(z_1+\omega_2,z_2)=\left(\frac{x_2^-}{x_2^+}\right)^Q\frac{x_1^--x_2^+}{x_1^--x_2^-}\frac{1-\frac{1}{x_1^+x_2^+}}{1-\frac{1}{x_1^+x_2^-}}\prod_{k=1}^{Q-1}G(M-Q+2k)\,, \tag{2.14}$$

$$\sigma^{QM}(z_1,z_2)\sigma^{QM}(z_1,z_2-\omega_2)=\left(\frac{x_1^+}{x_1^-}\right)^M\frac{x_1^--x_2^+}{x_1^--x_2^-}\frac{1-\frac{1}{x_1^+x_2^+}}{1-\frac{1}{x_1^+x_2^-}}\prod_{k=1}^{Q-1}G(M-Q+2k)\,, \tag{2.14}$$

where the following function was introduced:

$$G(\ell)=\frac{u_1-u_2-\frac{\mathrm{i}}{g}\ell}{u_1-u_2+\frac{\mathrm{i}}{g}\ell}.$$

[5] The second formula in (2.14) follows from the first one by using the unitarity condition $\sigma^{QM}(z_1,z_2)\sigma^{MQ}(z_2,z_1)=1$, and the identity $\prod_{k=0}^{M-1}G(Q-M+2k)=\prod_{k=1}^{Q-1}G(M-Q+2k)$.

Here for a Q-particle bound state, the variable $u \equiv u^Q$ is defined as

$$u^Q = x^+ + \frac{1}{x^+} - \frac{\mathrm{i}}{g}Q = x^- + \frac{1}{x^-} + \frac{\mathrm{i}}{g}Q.$$

Taking the logarithm of both sides of equations (2.14), one obtains the corresponding equations for the dressing phase. As for the case of fundamental particles, the dressing phase describing scattering of bound states is a well-defined function in the particle region of the z-torus and it should be analytically continued to the whole torus in a way compatible with crossing equations (2.14).

3. Φ- and Ψ-functions

In this section, we introduce Φ- and Ψ-functions and discuss the analytic continuation of χ in terms of these functions for x_1, x_2 close to the unit circle.

3.1. Φ-function

Let us introduce a function $\Phi(x_1, x_2)$ defined as the following double integral of the Cauchy type:

$$\Phi(x_1,x_2) = \mathrm{i}\oint \frac{\mathrm{d}w_1}{2\pi\,\mathrm{i}}\oint \frac{\mathrm{d}w_2}{2\pi\,\mathrm{i}}\frac{1}{(w_1-x_1)(w_2-x_2)}\log\frac{\Gamma\big[1+\frac{\mathrm{i}}{2}g\big(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\big)\big]}{\Gamma\big[1-\frac{\mathrm{i}}{2}g\big(w_1+\frac{1}{w_1}-w_2-\frac{1}{w_2}\big)\big]}, \tag{3.1}$$

where the integrals are over the unit circles. Since the integrand is a continuous function of ϑ_1 , ϑ_2 where $w_i = \mathrm{e}^{\mathrm{i}\vartheta_i}$, the function Φ is unambiguously defined by this integral formula for all values of x_1, x_2 not lying on the unit circle.

If x_1 (or x_2 or both) is on the unit circle, we can define $\Phi(\mathrm{e}^{\mathrm{i}\varphi_1}, x_2)$ as the following limit:

$$\Phi(\mathrm{e}^{\mathrm{i}\varphi_1}, x_2) \equiv \lim_{\epsilon\to 0^+}\Phi(\mathrm{e}^{\epsilon}\mathrm{e}^{\mathrm{i}\varphi_1}, x_2), \tag{3.2}$$

i.e. we approach the value $x_1 = \mathrm{e}^{\mathrm{i}\varphi_1}$ from exterior of the circle. This limit exists because for $|x_1| > 1$, $|x_2| \neq 1$ and $|w_2| = 1$, the integrand in (3.1) is a holomorphic function of w_1 in the annulus $\sqrt{1+\frac{1}{g^2}} - \frac{1}{g} < |w_1| < |x_1|$.

If we would define the limiting value of Φ by approaching $x_1 = \mathrm{e}^{\mathrm{i}\varphi_1}$ from the interior of the circle, we would obtain the result different from (3.2) and, for this reason, the function Φ is not continuous across $|x_1| = 1$. In fact, it is not difficult to see that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\Phi(\mathrm{e}^{-\epsilon}\mathrm{e}^{\mathrm{i}\varphi_1}, x_2) &= \Phi(\mathrm{e}^{\mathrm{i}\varphi_1}, x_2)\\ &\quad + \mathrm{i}\oint\frac{\mathrm{d}w}{2\pi\,\mathrm{i}}\frac{1}{w-x_2}\log\frac{\Gamma\big[1+\frac{\mathrm{i}}{2}g\big(2\cos\varphi_1 - w - \frac{1}{w}\big)\big]}{\Gamma\big[1-\frac{\mathrm{i}}{2}g\big(2\cos\varphi_1 - w - \frac{1}{w}\big)\big]}.\end{aligned} \tag{3.3}$$

The function Φ is skew-symmetric $\Phi(x_1, x_2) = -\Phi(x_2, x_1)$, and it also satisfies the following important relation:

$$\Phi\left(\frac{1}{x_1}, x_2\right) + \Phi(x_1, x_2) = \Phi(0, x_2)\,, \qquad |x_1| \neq 1\,, \tag{3.4}$$

which will be used to prove the crossing equations for the dressing phase. If $|x_1| = 1$, this relation is modified in an obvious way due to equation (3.3).

3.2. Ψ-function

Formula (3.3) suggests to introduce the following function:

$$\Psi(x_1,x_2)=\mathrm{i}\oint\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{1}{w-x_2}\log\frac{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(x_1+\frac{1}{x_1}-w-\frac{1}{w}\right)\right]}{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(x_1+\frac{1}{x_1}-w-\frac{1}{w}\right)\right]}\,, \tag{3.5}$$

which is equal to the second term in (3.3) for $x_1 = \mathrm{e}^{\mathrm{i}\varphi_1}$. By construction, $\Psi(x_1, x_2) = \Psi(1/x_1, x_2)$. If x_1 is close enough to the unit circle then the function $\Psi(x_1, x_2)$ is well-defined for all $|x_2| \neq 1$, and if $|x_2| = 1$ we obviously can use the same prescription (3.2), i.e. we approach $x_2 = \mathrm{e}^{\mathrm{i}\varphi_2}$ from the exterior of the unit circle. This representation for Ψ will apparently break down for such values of x_1 for which the arguments of the Γ-functions in (3.5) become negative integers for some w on the integration contour.

To analyze this situation in the case $|x_2| > 1$, it is convenient to integrate (3.5) by parts obtaining the following expression:

$$\begin{aligned}\Psi(x_1,x_2)=&-\frac{g}{2}\oint\frac{\mathrm{d}w}{2\pi\mathrm{i}}\log(w-x_2)\left(1-\frac{1}{w^2}\right)\\&\times\left[\psi\left(1+\frac{\mathrm{i}}{2}g\left(x_1+\frac{1}{x_1}-w-\frac{1}{w}\right)\right)+\psi\left(1-\frac{\mathrm{i}}{2}g\left(x_1+\frac{1}{x_1}-w-\frac{1}{w}\right)\right)\right],\end{aligned} \tag{3.6}$$

where the cut of the log-function should not intersect the unit circle[6]. Then, instead of dealing with the cuts of log Γ-functions, we analyze the location of poles of the ψ-functions, cf [34].

We use formula (3.6) as the definition of the function $\Psi(x_1, x_2)$ for $|x_2| > 1$ and for all values of x_1 where the integral representation is well defined. This defines Ψ as an analytic function on the x_1-plane with cuts. To determine the location of the cuts, we first notice that for a generic value of x_1, none of the infinitely many poles of the ψ-functions in the w-plane falls on the unit circle. If we now start continuously changing x_1, the poles start to move. At a certain point, x_1 might become such that two poles of one of the ψ-functions in equation (3.6) reach the circle. The values of x_1 for which this happens are solutions to the following equations:

$$x_1+\frac{1}{x_1}-w-\frac{1}{w}=\frac{2\mathrm{i}}{g}n,\qquad n\geqslant 1, \tag{3.7}$$

$$w+\frac{1}{w}-x_1-\frac{1}{x_1}=\frac{2\mathrm{i}}{g}n\,,\qquad n\geqslant 1\,, \tag{3.8}$$

where w is on the unit circle: $w = \mathrm{e}^{\mathrm{i}\theta}$. Solutions of (3.7) and (3.8) correspond to poles of the first and second ψ-functions in (3.6), respectively. It is clear that if x_1 and θ solve one of these equations then x_1 and $-\theta$ also do. Thus, for each x_1 there are two values of θ, and two poles of one of the ψ-functions are on the unit circle.

Equations (3.7) and (3.8) can be solved in terms of the following function $x(u)$:

$$x(u)=\frac{u}{2}\left(1+\sqrt{1-\frac{4}{u^2}}\right)\,, \tag{3.9}$$

where the cut of $x(u)$ is from −2 to 2. The function $x(u)$ maps the complex u-plane with the cut [−2, 2] onto the exterior of the unit circle, i.e. $|x(u)| > 1$ for any u. We are interested in

[6] In Mathematica one could use

$$\log(w-x)\to\mathrm{i}\tan^{-1}\left(\frac{w-\mathrm{Re}(x)}{\mathrm{Im}(x)}\right)+\frac{1}{2}\log(w^2-2\mathrm{Re}(x)w+\mathrm{Im}(x)^2+\mathrm{Re}(x)^2)$$

that gives (3.5) up to a constant which drops out of the full dressing phase.

solutions with $|x_1| < 1$ because we will be using the Ψ-function to define the dressing phase in this region. Then the solutions to (3.7) and (3.8) take the following form:

$$\mathrm{x}_\pm^{(n)}(u) = \frac{1}{x\left(u \pm \frac{2\mathrm{i}}{g}n\right)}, \quad u = w + \frac{1}{w}, \quad -2 \leqslant u \leqslant 2, \tag{3.10}$$

where the + sign is for the solution to equation (3.7). It is clear that all solutions to equations (3.7) and (3.8) lie in the lower and upper half-circles, respectively, because $\mathrm{Im}\, x(u + \mathrm{i}y) > 0$ and $\mathrm{Im}\, x(u - \mathrm{i}y) < 0$ for $y > 0$.

Further, we point out that equations (3.7) and (3.8) can be thought of as the constraint equations for $x^{n\pm}$ parameters of n-particle bound states, and the general solutions to these equations can be written as

$$\mathrm{x}_+^{(n)} = x^{n+}(z), \quad w = x^{n-}(z) = \mathrm{e}^{\mathrm{i}\theta}, \tag{3.11}$$

$$\mathrm{x}_-^{(n)} = x^{n-}(z), \quad w = x^{n+}(z) = \mathrm{e}^{\mathrm{i}\theta}. \tag{3.12}$$

The function Ψ is obviously discontinuous across any curve $\mathrm{x}_\pm^{(n)}$. However, for $|w| = 1$, the curves $\mathrm{x}_\pm^{(n)}$ are not closed[7] in the x-plane and, therefore, one can reach any point inside the unit circle without crossing them, see figure 2. Thus, the curves $\mathrm{x}_\pm^{(n)}$ represent the cuts of Ψ. In general, one should think of Ψ as being an analytic function (for $|x_2| > 1$) on an infinite genus surface. Specifying the cut structure, as described above, defines its particular branch that we will be using in this paper.

The jump across any of the cuts can be found by taking into account that for any positive integer n,

$$\psi(z - n) = -\frac{1}{z} + \text{regular terms}.$$

Then, enclosing the poles which are approaching the unit circle, one finds that the difference between the values of Ψ on the different edges of the cuts $\mathrm{x}_\pm^{(n)}$ is given by

$$\lim_{\epsilon\to 0^+} \left[\Psi\left(\mathrm{e}^{\epsilon}\, \mathrm{x}_+^{(n)}, x_2\right) - \Psi\left(\mathrm{e}^{-\epsilon}\, \mathrm{x}_+^{(n)}, x_2\right)\right] = \lim_{\epsilon\to 0^+} \frac{1}{\mathrm{i}} \log \frac{w\left(\mathrm{e}^{-\epsilon}\, \mathrm{x}_+^{(n)}\right) - x_2}{\frac{1}{w\left(\mathrm{e}^{-\epsilon}\, \mathrm{x}_+^{(n)}\right)} - x_2}, \tag{3.13}$$

$$\lim_{\epsilon\to 0^+} \left[\Psi\left(\mathrm{e}^{\epsilon}\, \mathrm{x}_-^{(n)}, x_2\right) - \Psi\left(\mathrm{e}^{-\epsilon}\, \mathrm{x}_-^{(n)}, x_2\right)\right] = \lim_{\epsilon\to 0^+} \frac{1}{\mathrm{i}} \log \frac{\frac{1}{w\left(\mathrm{e}^{-\epsilon}\mathrm{x}_-^{(n)}\right)} - x_2}{w\left(\mathrm{e}^{-\epsilon}\mathrm{x}_-^{(n)}\right) - x_2}. \tag{3.14}$$

Here $w(x_1)$ satisfies $|w(x_1)| < 1$ and solves the equation $x_1 + \frac{1}{x_1} - w - \frac{1}{w} = \pm\frac{2\mathrm{i}}{g}n$, where '+' sign is for equation (3.13). In deriving the formulas above we have used that $\lim_{\epsilon\to 0^+} w(\mathrm{e}^{-\epsilon}\, \mathrm{x}_\pm^{(n)})w(\mathrm{e}^{\epsilon}\, \mathrm{x}_\pm^{(n)}) = 1$ if $\mathrm{x}_\pm^{(n)}$ solves equations (3.7) and (3.8).

To treat the function Ψ for $|x_2| < 1$, we use the following identity:

$$\Psi(x_1, x_2) = -\Psi\left(x_1, \frac{1}{x_2}\right) + \Psi(x_1, 0), \quad |x_2| \neq 1. \tag{3.15}$$

Integrating the first term on the right-hand side of this equation by parts, we represent it in the form (3.6). The second term is defined by (3.5) for all values of x_1 except those lying on the curves $\mathrm{x}_\pm^{(n)}$ across which the function $\Psi(x_1, 0)$ is discontinuous. For a given x_1, we also have to choose cuts of the integrand in such a way that they do not intersect the unit circle. This is always possible because the position of the branch points of the integrand coincides with

[7] It is worth stressing that the unit circle covers twice any of the curves $\mathrm{x}_\pm^{(n)}$.

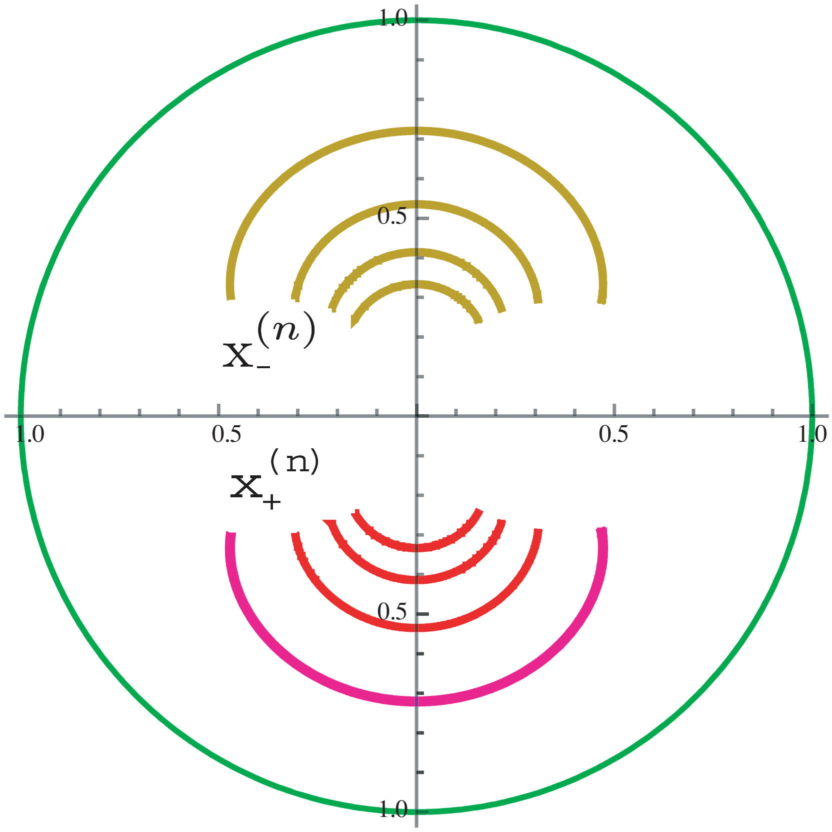


Figure 2. The curves $x_\pm^{(n)} = 1/x(u \pm \frac{2i}{g} n)$ with $-2 \leqslant u = w + \frac{1}{w} \leqslant 2$ for $g = 3$ and $n = 1, 2, 3, 4$. The endpoints of the curves correspond to $w = \pm 1$. The curves closest to the circle correspond to $n = 1$.

the position of the poles of the ψ-functions discussed above, and for each n these points come in pairs—one pair is inside the circle and another one is outside. For our purpose of defining the dressing factor, it does not matter how the cuts are chosen because different choices will lead to functions differing by an integer multiple of 2π, the latter drops out from the dressing factor.

To find the jump discontinuity of $\Psi(x_1, 0)$ across the cuts $x_\pm^{(n)}$, it is convenient to first differentiate it with respect to x_1:

$$\frac{d\Psi(x_1, 0)}{dx_1} = \Psi'(x_1, 0) = -\frac{g}{2}\left(1 - \frac{1}{x_1^2}\right)\oint \frac{dw}{2\pi\, i}\frac{1}{w} \times \left[\psi\left(1 + \frac{i}{2}g\left(x_1 + \frac{1}{x_1} - w - \frac{1}{w}\right)\right) + \psi\left(1 - \frac{i}{2}g\left(x_1 + \frac{1}{x_1} - w - \frac{1}{w}\right)\right)\right]. \tag{3.16}$$

Then, the computation of the difference between the values of $\Psi'(x_1, 0)$ on the different edges of the cuts $x_\pm^{(n)}$ follows the consideration above, and is given by

$$\lim_{\epsilon\to 0^+}\left[\Psi'\left(e^{\epsilon}\, x_+^{(n)}, 0\right) - \Psi'\left(e^{-\epsilon}\, x_+^{(n)}, 0\right)\right] = \frac{2}{i}\lim_{\epsilon\to 0^+}\frac{1 - \frac{1}{\left(e^{-\epsilon}\, x_+^{(n)}\right)^2}}{1 - \frac{1}{w\left(e^{-\epsilon}\, x_+^{(n)}\right)^2}}\frac{1}{w\left(e^{-\epsilon}\, x_+^{(n)}\right)}, \tag{3.17}$$

$$\lim_{\epsilon\to 0^+}\left[\Psi'\left(e^{\epsilon}x_-^{(n)},0\right)-\Psi'\left(e^{-\epsilon}x_-^{(n)},0\right)\right]=-\frac{2}{i}\lim_{\epsilon\to 0^+}\frac{1-\frac{1}{\left(e^{-\epsilon}x_-^{(n)}\right)^2}}{1-\frac{1}{w\left(e^{-\epsilon}x_-^{(n)}\right)^2}}\frac{1}{w\left(e^{-\epsilon}x_-^{(n)}\right)}. \tag{3.18}$$

Taking into account that

$$\frac{1-\frac{1}{x^2}}{1-\frac{1}{w(x)^2}}=\frac{dw(x)}{dx}, \tag{3.19}$$

we get the jump discontinuity of $\Psi(x_1,0)$

$$\lim_{\epsilon\to 0^+}\left[\Psi\left(e^{\epsilon}\,x_+^{(n)},0\right)-\Psi\left(e^{-\epsilon}\,x_+^{(n)},0\right)\right]=\frac{2}{i}\lim_{\epsilon\to 0^+}\log w\left(e^{-\epsilon}\,x_+^{(n)}\right), \tag{3.20}$$

$$\lim_{\epsilon\to 0^+}\left[\Psi\left(e^{\epsilon}x_-^{(n)},0\right)-\Psi\left(e^{-\epsilon}x_-^{(n)},0\right)\right]=\frac{2}{i}\lim_{\epsilon\to 0^+}\log\frac{1}{w\left(e^{-\epsilon}x_-^{(n)}\right)}, \tag{3.21}$$

where the integration constant is fixed from the requirement that the discontinuity vanishes at the end-points of the cuts (up to an integer multiple of 2π).

Combining these formulas with (3.13) and (3.14), one can easily check that the jump discontinuity of $\Psi(x_1,x_2)$ for $|x_2|<1$ is again given by the same formulas (3.13) and (3.14) up to an unimportant integer multiple of 2π.

3.3. *Function χ for $|x_1|\approx 1$, $|x_2|\approx 1$*

For $|x_1|>1$, $|x_2|>1$, the DHM integral representation for the function χ coincides with the Φ-function

$$\chi(x_1,x_2)=\Phi(x_1,x_2). \tag{3.22}$$

We want to use the DHM integral representation and the Φ- and Ψ-functions to fix the principal branch of the dressing phase $\theta(z_1,z_2)$, that is to define θ for any z_1,z_2 on the z-torus. It can be done in infinitely many ways because the phase is a function on the direct product of two infinite-genus Riemann surfaces. The only requirement we will impose is that for z_1,z_2 being on the principal branch, the dressing phase should satisfy the crossing equations with the function h given by (2.10).

For the values of x_1,x_2 close enough to the unit circle, the analytic continuation of the function χ is in fact unambiguous at least for $|x_{1,2}|>r_{cr}$. For instance, to determine χ for $|x_1|<1$, we can deform the integration contour of w_1, drag the point x_1 a little bit inside the unit circle and then enclose the pole at $w_1=x_1$, see figure 3. As a result, we obtain the following representation for χ:

$$\chi(x_1,x_2)=\Phi(x_1,x_2)-\Psi(x_1,x_2), \tag{3.23}$$

which is valid at least in the region $1>|x_1|>\sqrt{1+\frac{1}{g^2}}-\frac{1}{g}$, $|x_2|\geqslant 1$.

Similarly, for $|x_2|<1$ with x_2 staying close to the unit circle, we can deform the integration contour of w_2 to enclose the pole at $w_2=x_2$. This leads to the following representation for χ:

$$\chi(x_1,x_2)=\Phi(x_1,x_2)+\Psi(x_2,x_1), \tag{3.24}$$

where $1>|x_2|>\sqrt{1+\frac{1}{g^2}}-\frac{1}{g}$, $|x_1|\geqslant 1$.

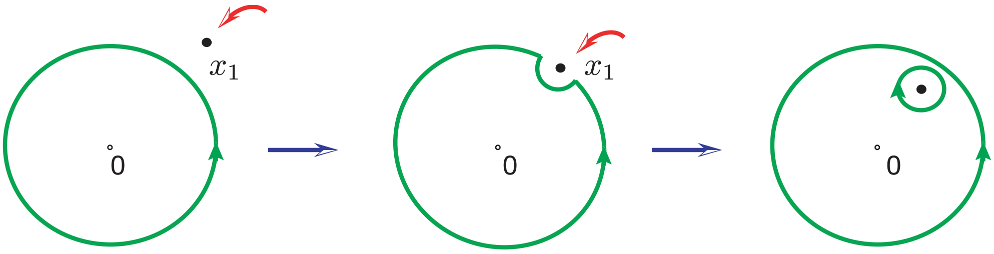


Figure 3. A little dragging of the variable x_1 inside the integration contour results into an extra contribution given by the integral around x_1 with integration performed in the clockwise direction.

Finally, if both $|x_1| < 1$, $|x_2| < 1$ and they are close to the unit circle, we can deform both integration contours and represent χ as follows:

$$\chi(x_1, x_2) = \Phi(x_1, x_2) - \Psi(x_1, x_2) + \Psi(x_2, x_1) + \mathrm{i}\log\frac{\Gamma\left[1 + \frac{\mathrm{i}}{2}g\left(x_1 + \frac{1}{x_1} - x_2 - \frac{1}{x_2}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2}g\left(x_1 + \frac{1}{x_1} - x_2 - \frac{1}{x_2}\right)\right]},$$

where the last term comes from the analytic continuation of (3.5) in x_2.

Since both functions Φ and Ψ are defined on the x_1- and x_2-planes with the cuts, the equations above define *a particular* analytic continuation of χ for all values of x_1 and x_2. It appears, however, that this continuation of χ is incompatible with the dressing phase considered as a function on the z-torus.

To understand this issue, we first notice that x_k in the functions χ appearing in the dressing phase (2.6) can be equal to either $x^+(z_k)$ or $x^-(z_k)$. Suppose we want to analytically continue the dressing phase $\theta(z_1, z_2)$ in the variable z_1 starting from a point inside the region $|x^{\pm}(z_1)| > 1$, see figure 1, to any other point on the z_1-torus. The question is whether one could choose such a path on the z_1-torus that its images $x^+(z_1)$ and $x^-(z_1)$ in the x^+- and x^--planes would not intersect any of the curves $\mathrm{x}_{\pm}^{(n)}$. If this were possible then equation (3.23) provided us with a well-defined analytic continuation of θ, and the images of the curves $\mathrm{x}_{\pm}^{(n)}$ on the z-torus would be its cuts. Clearly, such an analytic continuation path does not exist only if the image of one of the curves coincides with a one-cycle of the z-torus, and therefore, it divides the torus into two parts. In particular, from equations (3.11) and (3.12) we see that for the case of fundamental particles this happens only for curves corresponding to $n = 1$ because in this case the solutions (3.11) and (3.12) are equivalent to conditions $|x^-| = 1$ and $|x^+| = 1$, respectively, that give one-cycles of the z-torus, see figure 1. In what follows we discuss this issue in detail and obtain analytic continuations of the dressing phases corresponding to both the fundamental particles and their bound states.

4. The dressing phase for fundamental particles

In this section, we determine the analytic continuation of the dressing phase for fundamental particles for any z_1, z_2 on the z-torus and use it to prove the corresponding crossing equations.

4.1. Analytic continuation of the dressing phase

The dressing factor is not a double-periodic function on the z-torus, and therefore one needs to define it in the product of two infinite strips $-\frac{\omega_1}{2} \leqslant \mathrm{Im}(z) \leqslant \frac{\omega_1}{2}$. Since in reality we are interested in the dressing factor $\sigma = \mathrm{e}^{\mathrm{i}\theta}$, we will not be specific about the branches of log-functions which appear in the continuation of χ. We assume here that both particles are in the fundamental representation of the centrally extended $\mathfrak{su}(2|2)$ superalgebra, and the corresponding parameters $x^\pm$ obey the constraint (2.2).

Each of the infinite strips is divided by the curves $|x^\pm| = 1$ into regions where $|x^\pm|$ is either greater or smaller than unity, see figure 1; the region with $|x^\pm| > 1$ containing the real z-axis and where the dressing phase is an analytic function of z_1 and z_2 was called the particle region. By shifting this region by ω_2 upward or downward, one gets the anti-particle regions with $|x^\pm| < 1$. By shifting the real z-axis by $\frac{\omega_2}{2}$ upward, one gets the symmetry axis of the neighboring region with $|x^+| < 1$, $|x^-| > 1$ that is also the line corresponding to the real momentum of a mirror particle.

In general, by shifting the real z-axis by an integer multiple of $\frac{\omega_2}{2}$ upward or downward, one gets the symmetry axis of one of these regions. Thus, it is natural to denote the corresponding region as $\mathcal{R}_n$, and the product of two regions as $\mathcal{R}_{m,n}$ where $\mathcal{R}_{0,0}$ is the product of two particle regions, and $m, n \in \mathbb{Z}$ are these integer multiples of $\frac{\omega_2}{2}$ referring to the first and second z-variable, respectively.

Since the dressing phase is antisymmetric, it is sufficient to consider only the regions $\mathcal{R}_{m,n}$ with $m \geqslant n$. Moreover, the crossing equations relate the dressing factors in the regions $\mathcal{R}_{m,n}$ and $\mathcal{R}_{m+2,n}$, and, therefore, starting from the region $\mathcal{R}_{0,0}$ it is enough to determine the analytic continuation of the dressing factor to the regions $\mathcal{R}_{1,0}$, $\mathcal{R}_{2,0}$, $\mathcal{R}_{1,1}$ and $\mathcal{R}_{2,1}$, and to prove the crossing equations for the regions $\mathcal{R}_{0,0} \leftrightarrow \mathcal{R}_{2,0}$ and $\mathcal{R}_{0,1} \leftrightarrow \mathcal{R}_{2,1}$. Then, unitarity together with crossing equations allows one to continue analytically the dressing factor to any region $\mathcal{R}_{m,n}$. However, since the region $\mathcal{R}_{1,1}$ contains the real momentum line of the mirror theory, we decided to analytically continue the dressing phase to the region $\mathcal{R}_{3,1}$ without appealing to the crossing equations and to check them for these regions $\mathcal{R}_{1,1} \leftrightarrow \mathcal{R}_{3,1}$.

Region $\mathcal{R}_{1,0}$*:* $\{z_1, z_2\} \in \mathcal{R}_{1,0} \implies |x_1^+| < 1\,, \; |x_1^-| > 1\,; \; |x_2^\pm| > 1$
First we discuss the analytic continuation from the particle region $\mathcal{R}_{0,0}$ to the region $\mathcal{R}_{1,0}$ where $|x_1^+| < 1\,, \; |x_1^-| > 1\,; \; |x_2^\pm| > 1$.

We want to understand which curves $\mathrm{x}_\pm^{(n)}$ are crossed when the point z_1 moves upward along a vertical line in the z-torus and enters the region $\mathcal{R}_{1,0}$, see figure 4. In this region, we need to analyze the functions $\chi(x_1^+, x_2^\pm)$ only. Since the integral representation for χ is well defined for $|x_1^+| > r_{\rm cr}$, the dressing phase is obviously a holomorphic function in the vicinity of the curve $|x_1^+| = 1$, and it can be evaluated thereby using (3.23). The dressing phase cannot be however holomorphic everywhere in $\mathcal{R}_{1,0}$, because the curves $\mathrm{x}_\pm^{(n)}$ have images in this region, and the function $\Psi(x_1^+, x_2^\pm)$ is discontinuous across the curves. The images of the curves $\mathrm{x}_\pm^{(n)}$ in $\mathcal{R}_{1,0}$ are the cuts of the dressing phase on the z-torus, their end-points being the branch points of the dressing phase.

A simple analysis reveals that no cuts are met in the intersection of the region $|x_1^+| < 1$, $|x_1^-| > 1$ with the region $\mathrm{Im}(x_1^\pm) < 0$, see also figure 4. Thus, the dressing phase is a holomorphic function in this intersection. The region $\mathrm{Im}(x_1^\pm) < 0$ also contains the line corresponding to the real momentum of a mirror particle. It was considered as a natural candidate for the region of the mirror theory in [9] because it contains one of the 2^{Q-1} solutions to the Q-particle bound state equations and is in one-to-one correspondence with the u-plane. A choice of the mirror region is not however unique and, in particular, the Q-particle bound state solution used in [45][8] does not fall in the region $\mathrm{Im}(x_1^\pm) < 0$. For this reason, we are reluctant to refer to the region $\mathrm{Im}(x_1^\pm) < 0$ as a mirror one. Still, the region $\mathrm{Im}(x_1^\pm) < 0$ seems to be special because the dressing factor is analytic

[8] Strictly speaking, in [45] the real momentum line of the mirror theory was obtained by shifting the real z-axis downward by $\frac{\omega_1}{2}$. Obviously, these two choices are related to each other by reflection.

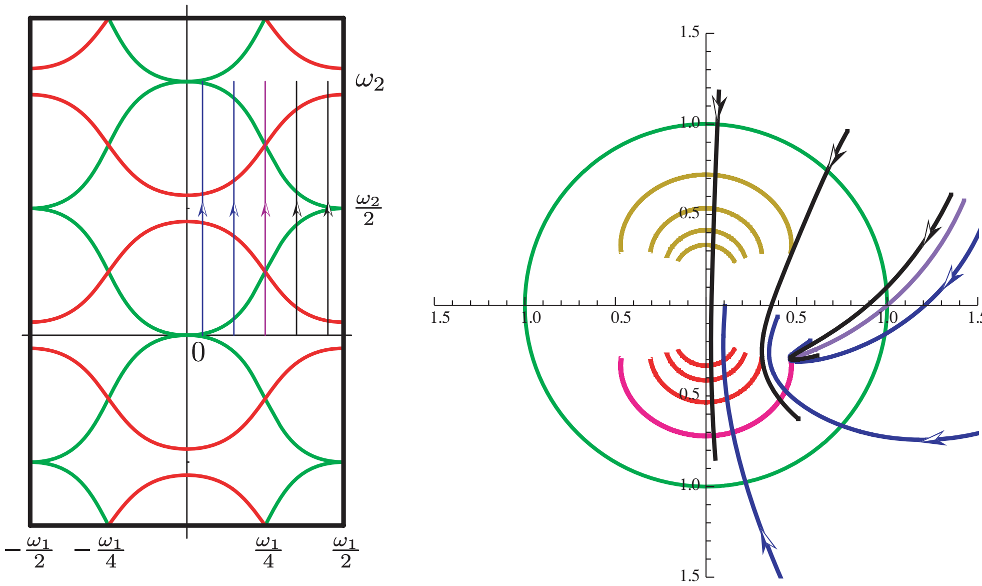


Figure 4. On the left figure analytic continuation paths are shown on the z-torus. On the right figure, blue curves represent the lines $x^+(z)$ corresponding to the torus variable z going upward from the real line to the line with $\mathrm{Im}(z) = \omega_2/\mathrm{i}$ and they have $|\mathrm{Re}(z)| \leqslant \frac{\omega_1}{4}$. Black curves $x^+(z)$ correspond to z going upward and have $|\mathrm{Re}(z)| \geqslant \frac{\omega_1}{4}$. Any black or blue curve intersects the lowest curve inside the circle. Paths sufficiently close to the lines $|\mathrm{Re}(z)| \leqslant \frac{\omega_1}{4}$ do not intersect any cut except the lowest curve and, therefore, they are used for analytic continuation.

there. This follows from the consideration above and from the fact that the dressing factor for fundamental particles is obviously analytic in the region $\mathcal{R}_{2,0}$ due to the crossing equation. We will return to the issue of non-uniqueness of bound state solutions in our conclusions.

Dragging z_1 upward, we observe that the first curve the point $x^+(z_1)$ reaches is $\mathrm{x}_+^{(1)} = x^+(z_1)$, $|x^-(z_1)| = 1$ that is the lower boundary of the anti-particle region $|x_1^+| < 1$, $|x_1^-| < 1$ and that is the image of the curve closest to the circle in the lower x^+-half-plane.

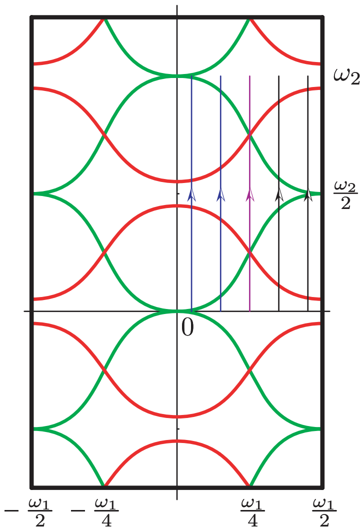
We conclude, therefore, that in the case of fundamental particles, the analytic branch of the functions $\chi(x_1^\pm, x_2^\pm)$ in the region $\mathcal{R}_{1,0}$ can be defined as

$$\mathcal{R}_{1,0}: \quad \chi\left(x_1^+, x_2^\pm\right) = \Phi\left(x_1^+, x_2^\pm\right) - \Psi\left(x_1^+, x_2^\pm\right), \tag{4.1}$$

$$\chi\left(x_1^-, x_2^\pm\right) = \Phi\left(x_1^-, x_2^\pm\right), \tag{4.2}$$

where Ψ is given by (3.6). Let us also mention that the region $\mathcal{R}_{-3,0}$ obtained by shifting the point z_1 downward also has $|x_1^+| < 1$, $|x_1^-| > 1$. The dressing phase, however, differs there from (4.1) by the double crossing term (2.11).

Region $\mathcal{R}_{2,0}$*:* $\{z_1, z_2\} \in \mathcal{R}_{2,0} \implies |x_1^+| < 1\,,\ |x_1^-| < 1\,;\ |x_2^\pm| > 1$
Dragging the point z_1 further upward into the anti-particle region $|x_1^\pm| < 1$, we must *inevitably* cross the curve $\mathrm{x}_+^{(1)}$, the latter maps to the one-cycle $|x_1^-| = 1$ of the z-torus. Then formula (3.23) for $\chi(x_1^+, x_2)$ should be modified because, as was discussed in the previous section, one pole of the first ψ-function in (3.6) moves outside the circle and another one moves inside.



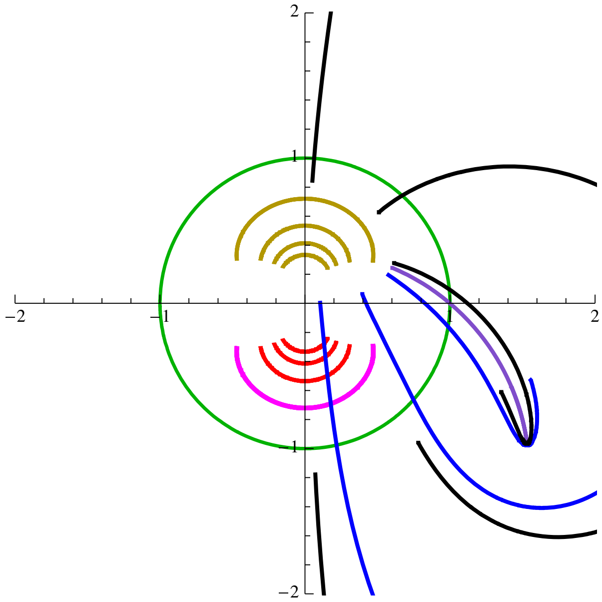


Figure 5. Blue and black curves on the right figure represent the curves $x^-(z)$ corresponding to the curves $x^+(z)$ in figure 4. No black curve intersects the cuts inside the circle, and the blue curves close enough to $\mathrm{Re}(z) = \frac{\omega_1}{4}$ do not intersect the cuts either.

Therefore, once the point z_1 crosses the lower boundary of the anti-particle region $|x_1^\pm| < 1$, we should add to Ψ the following term:

$$\frac{1}{\mathrm{i}} \log \frac{w(x_1^+) - x_2^\pm}{\frac{1}{w(x_1^+)} - x_2^\pm}, \tag{4.3}$$

where we have taken into account formula (3.13) for the jump discontinuity of the Ψ-function. Here $w(x_1^+)$ solves the equation $x_1^+ + \frac{1}{x_1^+} - w - \frac{1}{w} = \frac{2\mathrm{i}}{g}$ and satisfies $|w(x_1^+)| < 1$. Since $|x_1^-| < 1$ once z_1 crosses the boundary, we conclude that $w(x_1^+) = x_1^-$. Thus, we get the following expressions for the functions χ with the first particle being in the anti-particle region $\mathcal{R}_{2,0}$,

$$\mathcal{R}_{2,0}: \quad \chi(x_1^+, x_2^\pm) = \Phi(x_1^+, x_2^\pm) - \Psi(x_1^+, x_2^\pm) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^\pm}{x_1^- - x_2^\pm}, \tag{4.4}$$

$$\chi(x_1^-, x_2^\pm) = \Phi(x_1^-, x_2^\pm) - \Psi(x_1^-, x_2^\pm), \tag{4.5}$$

where $|x_1^+| < 1$, $|x_1^-| < 1$ and $|x_2^\pm| > 1$, and the functions Φ and Ψ are given by (3.1) and (3.6) for all values of $x_1^\pm$ from the region. Let us stress that in the z_1-plane the anti-particle region is obtained from the particle region containing the real z-axis by shifting it upward by ω_2.

These formulas define a certain analytic branch of the functions $\chi(x_1^\pm, x_2^\pm)$ in $\mathcal{R}_{2,0}$ and, as will be discussed in the next subsection, they are sufficient to prove the crossing equation for z_1, z_2 being in the particle region. It is worth stressing that even though the functions χ are not analytic in the region $\mathcal{R}_{2,0}$ because of the cuts located in the intersection of the anti-particle region with the regions $\mathrm{Im}(x^\pm) < 0$ and $\mathrm{Im}(x^\pm) > 0$, as is evident from figures 4 and 5, the dressing factor itself is analytic in $\mathcal{R}_{2,0}$ because it is related to the dressing factor in $\mathcal{R}_{0,0}$ by the crossing equation.

We also point out that the functions χ in (4.4) are holomorphic in the vicinity of the lower boundary of the anti-particle region because these representations were derived by deforming the integration contour in (3.1) and (3.6). In other words, the curve $|x^-(z)| = 1$ is not a cut of the dressing phase[9] but rather the boundary of validity of the integral representation for χ. Crossing this curve enforces a modification of the integral representation, as described above.

In our treatment above we have chosen a path for analytic continuation of the dressing phase by starting with the variable z_1 from the particle region and dragging it upward from the real axes. Analogously, we could consider a path of different orientation, i.e. the one which is obtained by shifting z_1 downward from the real axis. The interested reader may consult the appendix, where the results of this analytic continuation are sketched. Now we find the analytic continuation for the dressing phase in the regions $\mathcal{R}_{k,1}, k = 2, 3,$. This is relevant for proving that the dressing phase of the mirror theory also satisfies the same crossing equations.

Region $\mathcal{R}_{1,1}$: $\{z_1, z_2\} \in \mathcal{R}_{1,1} \quad\Longrightarrow\quad |x_1^+| < 1\,,\ |x_1^-| > 1\,;\ |x_2^+| < 1\,,\ |x_2^-| > 1$
Consider the case where the particles are in the region $\mathcal{R}_{1,1}$ with $|x_k^+| < 1, |x_k^-| > 1$ obtained from the region $\mathcal{R}_{0,0}$ by moving both points z_1 and z_2 upward. If the second particle is in the region $|x_2^\pm| > 1$ the functions χ are given by (4.1) and (4.2). If the point z_2 is shifted upward, then we should add the extra contributions coming from Φ and Ψ functions, and we derive the following expressions for the functions χ:

$$
\begin{aligned}
\mathcal{R}_{1,1}:\quad \chi\left(x_1^+, x_2^+\right) &= \Phi\left(x_1^+, x_2^+\right) + \Psi\left(x_2^+, x_1^+\right) - \Psi\left(x_1^+, x_2^+\right)\\
&\quad + \mathrm{i}\log\frac{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}\,,\\
\chi\left(x_1^+, x_2^-\right) &= \Phi\left(x_1^+, x_2^-\right) - \Psi\left(x_1^+, x_2^-\right),\\
\chi\left(x_1^-, x_2^+\right) &= \Phi\left(x_1^-, x_2^+\right) + \Psi\left(x_2^+, x_1^-\right),\\
\chi\left(x_1^-, x_2^-\right) &= \Phi\left(x_1^-, x_2^-\right),
\end{aligned}
\tag{4.6}
$$

where the last term in the formula for $\chi(x_1^+, x_2^+)$ comes from the analytic continuation of $\Psi(x_1^+, x_2^+)$ in z_2 shifted upward.

Region $\mathcal{R}_{2,1}$: $\{z_1, z_2\} \in \mathcal{R}_{2,1} \quad\Longrightarrow\quad |x_1^+| < 1\,,\ |x_1^-| < 1\,;\ |x_2^+| < 1\,,\ |x_2^-| > 1$
Shifting z_1 further upward into the anti-particle region, we get

$$
\begin{aligned}
\mathcal{R}_{2,1}:\quad \chi\left(x_1^+, x_2^+\right) &= \Phi\left(x_1^+, x_2^+\right) + \Psi\left(x_2^+, x_1^+\right) - \Psi(x_1^+, x_2^+) + \frac{1}{\mathrm{i}}\log\frac{\frac{1}{x_1^-} - x_2^+}{x_1^- - x_2^+}\\
&\quad + \mathrm{i}\,\log\frac{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]},\\
\chi\left(x_1^+, x_2^-\right) &= \Phi(x_1^+, x_2^-) - \Psi\left(x_1^+, x_2^-\right) + \frac{1}{\mathrm{i}}\log\frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-},\\
\chi\left(x_1^-, x_2^+\right) &= \Phi\left(x_1^-, x_2^+\right) - \Psi\left(x_1^-, x_2^+\right) + \Psi\left(x_2^+, x_1^-\right)\\
&\quad + \mathrm{i}\log\frac{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]},\\
\chi\left(x_1^-, x_2^-\right) &= \Phi\left(x_1^-, x_2^-\right) - \Psi\left(x_1^-, x_2^-\right).
\end{aligned}
\tag{4.7}
$$

[9] It cannot be a cut already for the reason that it coincides with the one-cycle of the torus.

Region $\mathcal{R}_{3,1}$*:* $\{z_1, z_2\} \in \mathcal{R}_{3,1} \quad \Longrightarrow \quad |x_1^+| > 1\,,\ |x_1^-| < 1\,;\ |x_2^+| < 1\,,\ |x_2^-| > 1$
Dragging the point z_1 further upward into the region $|x_1^-| < 1$, $|x_1^+| > 1$, we cross the curve $\mathrm{x}_-^{(1)}$ that is mapped to the curve $|x_1^+| = 1$ on the z-torus, the latter being the upper boundary of the anti-particle region.

Then x_1^+ goes outside the unit circle and we need to drop the Ψ-function from (4.4), and x_1^- crosses $\mathrm{x}_-^{(1)}$ and produces an extra contribution to (4.7) because one pole of the second ψ-function in (3.6) moves outside the circle and another one moves inside. The result of the analytic continuation is then given by

$$\mathcal{R}_{3,1}: \quad \chi\left(x_1^+, x_2^+\right) = \Phi\left(x_1^+, x_2^+\right) + \Psi\left(x_2^+, x_1^+\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^+}{x_1^- - x_2^+}\,,$$

$$\chi\left(x_1^+, x_2^-\right) = \Phi\left(x_1^+, x_2^-\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-}\,,$$

$$\begin{aligned}\chi\left(x_1^-, x_2^+\right) = {} & \Phi\left(x_1^-, x_2^+\right) - \Psi\left(x_1^-, x_2^+\right) + \Psi\left(x_2^+, x_1^-\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^+} - x_2^+}{x_1^+ - x_2^+} \\ & + \mathrm{i} \log \frac{\Gamma\left[1 + \frac{\mathrm{i}}{2} g \left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2} g \left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}\,,\end{aligned}$$

$$\chi\left(x_1^-, x_2^-\right) = \Phi\left(x_1^-, x_2^-\right) - \Psi\left(x_1^-, x_2^-\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^+} - x_2^-}{x_1^+ - x_2^-}\,. \tag{4.8}$$

Now we have all the necessary ingredients to verify the fulfillment of the crossing equations for the dressing phase of fundamental particles.

4.2. *The crossing equations for fundamental particles*

Having obtained the dressing phase as an analytic function with cuts on the product of two infinite strips $-\frac{\omega_1}{2} \leqslant \mathrm{Im}(z) \leqslant \frac{\omega_1}{2}$, we can now evaluate the left-hand side of the crossing equation

$$\Delta\theta \equiv \theta(z_1, z_2) + \theta(z_1 + \omega_2, z_2).$$

In terms of the χ-functions, $\Delta\theta$ takes the form

$$\begin{aligned}\Delta\theta = {} & \chi\left(x_1^+, x_2^+\right) - \chi\left(x_1^+, x_2^-\right) - \chi\left(x_1^-, x_2^+\right) + \chi\left(x_1^-, x_2^-\right) \\ & + \chi\left(1/x_1^+, x_2^+\right) - \chi\left(1/x_1^+, x_2^-\right) - \chi\left(1/x_1^-, x_2^+\right) + \chi\left(1/x_1^-, x_2^-\right).\end{aligned} \tag{4.9}$$

Considered as the function on two z-planes, the crossing equation must hold for any choice of the pair $\{z_1, z_2\}$. In what follows we will restrict ourselves to checking the crossing equation for two different cases, namely for $\{z_1, z_2\} \in \mathcal{R}_{0,0}$ and $\{z_1, z_2\} \in \mathcal{R}_{1,1}$. We recall that these cases correspond to both z_1 and z_2 being in the particle region or in the region relevant to the mirror theory, respectively.

We start with the case $\{z_1, z_2\} \in \mathcal{R}_{0,0} \Rightarrow |x_1^\pm| > 1\,;\ |x_2^\pm| > 1$. Then, $|1/x^\pm| < 1$ and, therefore, the arguments of the χ-functions occurring in the second line of equation (4.9) are in the region $\mathcal{R}_{2,0}$. Thus, evaluating the second line in equation (4.9), we have to use formulas (4.4) and (4.5) with the substitution in the latter $x_1^\pm \to 1/x_1^\pm$. Taking into account the identity (3.4), one finds that the contribution of Φ-functions in $\Delta\theta$ cancels out and one gets

$$\Delta\theta = \Psi\left(\frac{1}{x_1^-}, x_2^+\right) - \Psi\left(\frac{1}{x_1^+}, x_2^+\right) + \Psi\left(\frac{1}{x_1^+}, x_2^-\right) - \Psi\left(\frac{1}{x_1^-}, x_2^-\right) + \frac{1}{i}\log\frac{x_1^- - x_2^+}{\frac{1}{x_1^-} - x_2^+}\frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-}. \tag{4.10}$$

The Ψ-function satisfies a number of important identities which are listed in Appendix A.2. In particular, by using formula (A.4) valid in the region $|x_1^\pm| > 1$ and $|x_2| > 1$, we get

$$\Psi\left(\frac{1}{x_1^-}, x_2^+\right) - \Psi\left(\frac{1}{x_1^+}, x_2^+\right) + \Psi\left(\frac{1}{x_1^+}, x_2^-\right) - \Psi\left(\frac{1}{x_1^-}, x_2^-\right) = \frac{1}{i}\log\frac{\left(1 - \frac{1}{x_1^- x_2^+}\right)\left(1 - \frac{1}{x_1^+ x_2^+}\right)}{\left(1 - \frac{1}{x_1^- x_2^-}\right)\left(1 - \frac{1}{x_1^+ x_2^-}\right)}.$$

Finally,

$$\Delta\theta = \frac{1}{i}\log\frac{\left(1 - \frac{1}{x_1^- x_2^+}\right)\left(1 - \frac{1}{x_1^+ x_2^+}\right)}{\left(1 - \frac{1}{x_1^- x_2^-}\right)\left(1 - \frac{1}{x_1^+ x_2^-}\right)} + \frac{1}{i}\log\frac{x_1^- - x_2^+}{\frac{1}{x_1^-} - x_2^+}\frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-} = \frac{1}{i}\log\left[\frac{x_2^-}{x_2^+}h(x_1, x_2)\right],$$

which is the correct crossing equation for the dressing phase of fundamental particles.

Now we would like to verify the crossing equations for the case $\{z_1, z_2\} \in \mathcal{R}_{1,1} \Rightarrow |x_1^+| < 1\,,\ |x_1^-| > 1\,;\ |x_2^+| < 1\,,\ |x_2^-| > 1$. Since $1/|x_1^+| > 1$ and $1/|x_1^-| < 1$, the arguments of the χ-functions in the second line of equation (4.9) are in the region $\mathcal{R}_{3,1}$. Thus, to evaluate the second line in equation (4.9), we have to use formulas (4.6) and (4.8) with the substitution in the latter $x_1^\pm \to 1/x_1^\pm$. With the account of identities (3.4) and (3.15), we get

$$\begin{aligned}\Delta\theta = {}& \Psi\left(\frac{1}{x_1^-}, x_2^+\right) - \Psi\left(\frac{1}{x_1^+}, x_2^+\right) + \Psi\left(\frac{1}{x_1^+}, x_2^-\right) - \Psi\left(\frac{1}{x_1^-}, x_2^-\right)\\ &+ \frac{1}{i}\log\frac{x_1^- - x_2^+}{\frac{1}{x_1^-} - x_2^+}\frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-} + \frac{1}{i}\log\frac{x_1^+ - x_2^-}{\frac{1}{x_1^+} - x_2^-}\frac{\frac{1}{x_1^+} - x_2^+}{x_1^+ - x_2^+}\\ &+ \frac{1}{i}\log\frac{\Gamma\left[1 + \frac{i}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 - \frac{i}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}\frac{\Gamma\left[1 - \frac{i}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 + \frac{i}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}.\end{aligned} \tag{4.11}$$

First, the ratio of the Γ-functions is simplified to

$$\begin{aligned}&\frac{1}{i}\log\frac{\Gamma\left[1 + \frac{i}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 - \frac{i}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}\frac{\Gamma\left[1 - \frac{i}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 + \frac{i}{2}g\left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}\\ &\quad= \frac{1}{i}\log\frac{g^2}{4}(x_1^- - x_2^+)(x_1^+ - x_2^+)\left(1 - \frac{1}{x_1^- x_2^+}\right)\left(1 - \frac{1}{x_1^+ x_2^+}\right).\end{aligned} \tag{4.12}$$

Second, by using the identities (A.5) and (A.10), one obtains

$$\Psi\left(\frac{1}{x_1^+}, x_2^-\right) - \Psi\left(\frac{1}{x_1^-}, x_2^-\right) = \frac{1}{i}\log\frac{x_2^-}{\left(\frac{1}{x_1^- x_2^-} - 1-\right)\left(x_1^+ - x_2^-\right)}, \tag{4.13}$$

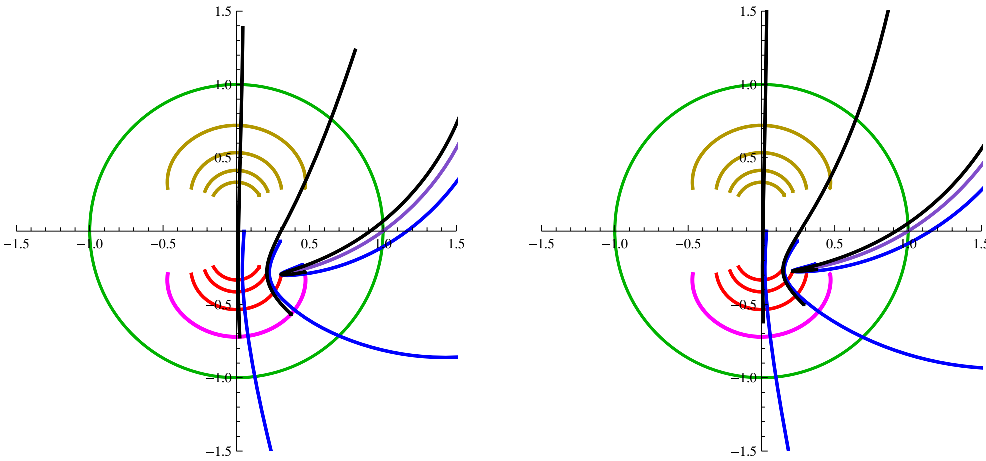


Figure 6. On the left picture blue curves represent $x^+(z)$ corresponding to z going upward from the real line of the z-torus of a two-particle bound state to the line with $\mathrm{Im}(z) = \omega_2/\mathrm{i}$ and have $|\mathrm{Re}(z)| \leqslant \frac{\omega_1}{4}$. Black curves $x^+(z)$ correspond to z going upward and they have $|\mathrm{Re}(z)| \geqslant \frac{\omega_1}{4}$. Any blue curve intersects the second lower curve in the circle. On the right picture a three-particle bound state is considered, and any blue curve intersects the third lower curve. Curves $x^-(z)$ are shown in figure 10.

$$\Psi\left(\frac{1}{x_1^-}, x_2^+\right) - \Psi\left(\frac{1}{x_1^+}, x_2^+\right) = \frac{1}{\mathrm{i}} \log \frac{4}{g^2} \frac{x_1^+}{x_2^+} \frac{1}{\left(x_2^+ - x_1^-\right)\left(x_1^+ - \frac{1}{x_2^+}\right)}. \qquad (4.14)$$

Substituting these results in equation (4.11), we again recover the crossing equation (2.8). This verification of the crossing equation for $\mathcal{R}_{1,1}$ confirms correctness of our analytic continuation of the dressing phase into this region.

5. Bound state dressing factor of string theory

In this section we discuss the analytic continuation of the dressing phase $\theta^{QM}(z_1, z_2)$ for the scattering matrix of Q-particle and M-particle bound states. Further, we use this continuation to prove the general crossing equations (2.14).

5.1. Analytic continuation of the bound state dressing phase

As was reviewed in section 2, in the particle region $|x^\pm| > 1$ and in terms of the variables $x^\pm$, the dressing phase θ^{QM} has the same functional form as the fundamental one. For this reason, it seems natural to use the analytic continuation described in the previous section also in the general case. It appears, however, that this continuation is incompatible with the crossing equations (2.14), which were derived from the ones for fundamental particles by using the fusion procedure.

In this subsection we identify the analytic continuation of the dressing phase θ^{QM} which leads to equations (2.14). We assume here that the first and the second particles are Q- and M-particle bound states, respectively, and that their kinematic parameters $x_1^\pm$ and $x_2^\pm$ obey the constraints (2.13).

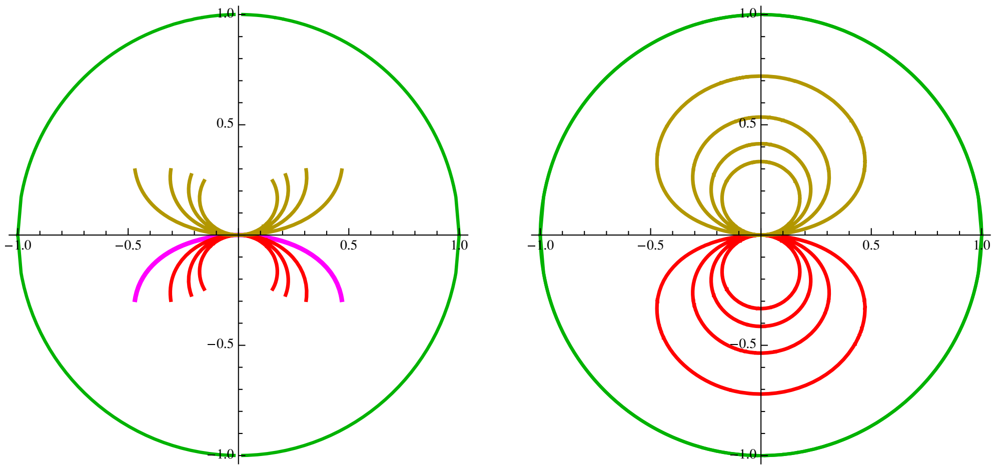


Figure 7. On the left picture the curves $\check{x}_\pm^{(n)} = 1/x(u \pm \frac{2i}{g}n)$ with $|u| \geqslant 2$ for $g = 3$ and $n = 1, 2, 3, 4$. The endpoints of the curves correspond to $u = \pm 2$. The curves closest to the real line correspond to $n = 1$. On the right picture the curves $1/x(u \pm \frac{2i}{g}n)$ with $|u| < \infty$ are shown.

Region $\mathcal{R}_{1,0}$*:* $\{z_1, z_2\} \in \mathcal{R}_{1,0} \quad \Longrightarrow \quad |x_1^+| < 1\,,\ |x_1^-| > 1\,;\ |x_2^\pm| > 1$
Again, we begin with defining the analytic continuation from the particle region $\mathcal{R}_{0,0}$ to the region $\mathcal{R}_{1,0}$ whose kinematic description is given above.

We first need to determine which curves $\mathrm{x}_\pm^{(n)}$ are crossed when the point z_1 moves upward along a vertical line in the z-torus and enters the region $\mathcal{R}_{1,0}$, see figure 6. Recall that in the case of fundamental particles, the point z_1 in the intersection of the region $\mathcal{R}_{1,0}$ with the region $\mathrm{Im}(x^\pm) < 0$ does not cross on its way the image of any curve $\mathrm{x}_\pm^{(n)}$, until it reaches the curve $|x_1^-| = 1$ that is the upper boundary of $\mathcal{R}_{1,0}$ and an image of $\mathrm{x}_+^{(1)}$. As a result, the dressing phase $\theta^{11} \equiv \theta$ appears to be a meromorphic function in this intersection. In the Q-particle case, the upper boundary of $\mathcal{R}_{1,0}$ is an image of the curve $\mathrm{x}_+^{(Q)}$, as is evident from equation (3.11), and there are images of the first $Q-1$ curves $\mathrm{x}_+^{(n)}$ in the intersection of $\mathcal{R}_{1,0}$ with the region $\mathrm{Im}(x^\pm) < 0$. This indicates that the dressing phase θ^{QM} should have cuts in the intersection. If one would choose the cuts to coincide with the images of $\mathrm{x}_+^{(n)}$, which therefore are not allowed to be crossed, then the analytic continuation of the dressing phase would obviously be the same as for fundamental particles. It turns out, however, that the crossing equations force us to analytically continue across the curves $\mathrm{x}_+^{(n)}$ and therefore, the cuts in the z-plane should be chosen complementary to the images of $\mathrm{x}_+^{(n)}$.

A convenient and natural choice of the cuts in the x- and z-planes is provided by equation (3.10). This equation suggests to identify the cuts with the curves $\check{x}_+^{(n)} = \frac{1}{x(u+\frac{2i}{g}n)}$, $n = 1, \ldots, Q-1$, but where the parameter u takes values in the region $|u| \geqslant 2$. In the x-plane all these curves go through the origin $x = 0$ corresponding to $u = \pm\infty$, see figure 7. In the z-plane the images of the curves are in the intersection of the region $\mathcal{R}_{1,0}$ with the region $\mathrm{Im}(x^\pm) < 0$, and the origin $x = 0$ corresponds to the points $z = -\frac{\omega_1}{2} + \frac{\omega_2}{2}$ and $z = \frac{\omega_1}{2} + \frac{\omega_2}{2}$ that are one and the same point on the z-torus.

Obviously, the union of the curves $x_+^{(n)}$ and $\check{x}_+^{(n)}$ is a closed curve in the x-plane, and for $n = 1, \ldots, Q-1$ its image in the region $\mathcal{R}_{1,0}$ is a one-cycle and divides the z-torus into two parts. Let us denote the corresponding curve in the z-torus as $\mathcal{X}_+^{(n)}$. Thus, the region $\mathcal{R}_{1,0}$ is divided by these curves into Q smaller regions, see figure 8. We denote the region bounded by the curves $\mathcal{X}_+^{(n-1)}$ and $\mathcal{X}_+^{(n)}$ as $\mathcal{R}_{1,0}^n$, where $n = 1, \ldots, Q$. The curve $\mathcal{X}_+^{(0)}$ is the lower

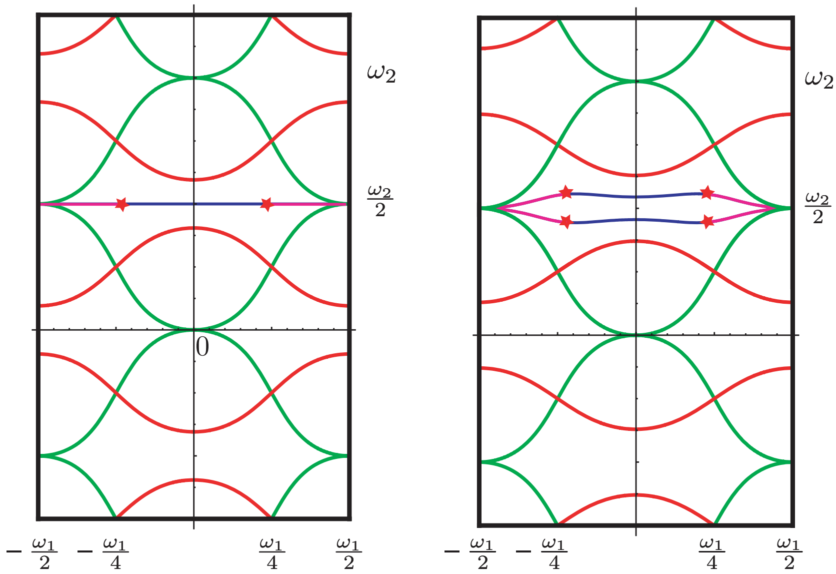


Figure 8. Division of $\mathcal{R}_1$ into smaller regions by curves $\chi_+^{(n)}$ is shown for two-particle (the left figure) and three-particle (the right figure) bound states. The cuts $\check{x}_+^{(n)}$ of the dressing phase are drawn in purple. In the two-particle case, the curve $\chi_+^{(1)}$ coincides with the real line of the mirror theory.

boundary of the region $\mathcal{R}_{1,0}$ with $|x_1^+| = 1$, while the curve $\mathcal{X}_+^{(Q)} = \mathrm{x}_+^{(Q)}$ is the upper boundary of $\mathcal{R}_{1,0}$ with $|x_1^-| = 1$.

Thus, to reach the region $\mathcal{R}_{1,0}^n$, one should analytically continue through the first $n-1$ curves $\mathrm{x}_+^{(n)}$. Therefore, in contradistinction to the case of fundamental particles, one gets $n-1$ extra contributions. As we will see, when properly combined, these extra contributions lead to the correct crossing equation.

We conclude, therefore, that in the Q-particle bound state case the images of the curves $\check{x}_+^{(n)}$, $n = 1, \ldots, Q-1$; $\mathrm{x}_+^{(n)}$, $n = Q+1, \ldots, \infty$; and $\mathrm{x}_-^{(n)}$, $n = 1, \ldots, \infty$ in the region $\mathcal{R}_{1,0}$ are the cuts of the dressing phase on the z-torus, see figure 9, and the analytic continuation of the functions $\chi(x_1^\pm, x_2^\pm)$ in the region $\mathcal{R}_{1,0}^n$ is given by

$$
\begin{aligned}
\mathcal{R}_{1,0}^n: \quad & \chi\left(x_1^+, x_2^\pm\right) = \Phi\left(x_1^+, x_2^\pm\right) - \Psi\left(x_1^+, x_2^\pm\right) - \frac{1}{\mathrm{i}} \log \prod_{j=1}^{n-1} \frac{w_j^-\left(x_1^+\right) - x_2^\pm}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^\pm}, \\
& \chi\left(x_1^-, x_2^\pm\right) = \Phi\left(x_1^-, x_2^\pm\right),
\end{aligned}
\tag{5.1}
$$

where $n = 1, \ldots, Q$. Here we have introduced the functions $w_j^\pm(x)$ defined as the solutions to the following equations:

$$
x + \frac{1}{x} - w_j^- - \frac{1}{w_j^-} = \frac{2\mathrm{i}}{g} j, \quad w_j^+ + \frac{1}{w_j^+} - x - \frac{1}{x} = \frac{2\mathrm{i}}{g} j, \quad \left|w_j^\pm\right| < 1. \tag{5.2}
$$

General solutions to these equations can be given in terms of the function $x(u)$ as follows

$$
w_j^-\left(x_1^+\right) = w_{Q-j}^+\left(x_1^-\right) = \frac{1}{x\left(u_1 + \frac{\mathrm{i}}{g}(Q-2j)\right)}, \tag{5.3}
$$

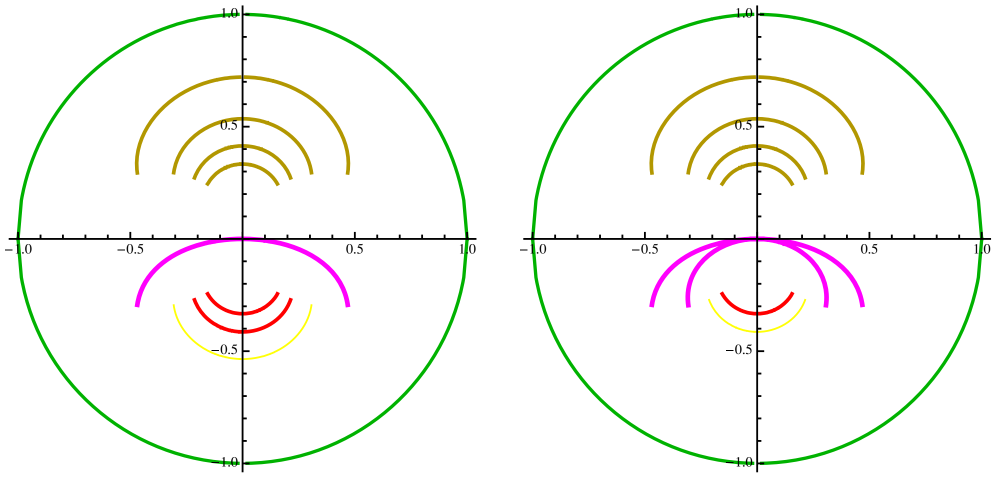


Figure 9. The left and right pictures represent the x-plane cuts of the two- and three-particle bound state dressing factors in the region $\mathcal{R}_{1,0}$, respectively. The yellow curves are $\mathrm{x}_+^{(Q)}$ with $Q = 2, 3$, and they are not cuts of the dressing factors.

where we introduce the u_1-plane variable

$$u_1 = x_1^+ + \frac{1}{x_1^+} - \frac{\mathrm{i}}{g}Q = x_1^- + \frac{1}{x_1^-} + \frac{\mathrm{i}}{g}Q. \tag{5.4}$$

In the u_1-plane the images of the curves $\check{x}_+^{(n)}$, $n = 1, \ldots, Q-1$; $\mathrm{x}_+^{(n)}$, $n = Q+1, \ldots, \infty$; and $\mathrm{x}_-^{(n)}$, $n = 1, \ldots, \infty$ are given by the following line segments:

$$\check{x}_+^{(n)}: \quad u_1 = u - \frac{\mathrm{i}}{g}(Q - 2n), \quad |u| \geqslant 2, \quad n = 1, \ldots, Q-1, \tag{5.5}$$

$$x_+^{(n)}: \quad u_1 = u - \frac{\mathrm{i}}{g}(Q - 2n), \quad |u| \leqslant 2, \quad n = Q+1, \ldots, \infty, \tag{5.6}$$

$$x_-^{(n)}: \quad u_1 = u + \frac{\mathrm{i}}{g}(Q - 2n), \quad |u| \leqslant 2, \quad n = 1, \ldots, \infty. \tag{5.7}$$

As already mentioned above, the images of the curves $\check{x}_+^{(n)}$, $n = 1, \ldots, Q-1$; $\mathrm{x}_+^{(n)}$, $n = Q+1, \ldots, \infty$; and $\mathrm{x}_-^{(n)}$, $n = 1, \ldots, \infty$ in the region $\mathcal{R}_{1,0}$ are the cuts of the dressing phase on the z-torus, see figures 8 and 9. We also point out that if $Q = 2m$ is even then the cut with $n = m$ falls on the real line of the mirror theory. Indeed, from equation (5.5) we see that for $\check{x}_+^{(m)}$ the parameter u_1 coincides with the real u obeying $|u| \geqslant 2$. We also see that for $Q = 2m$ the curve $\mathcal{X}_+^{(m)}$ is the real line of the mirror theory.

Region $\mathcal{R}_{2,0}$*:* $\{z_1, z_2\} \in \mathcal{R}_{2,0} \implies |x_1^+| < 1,\ |x_1^-| < 1;\ |x_2^\pm| > 1$
Moving the point z_1 from the region $\mathcal{R}_{1,0}^Q$ further upward into the anti-particle region $\mathcal{R}_{2,0}$ with $|x_1^\pm| < 1$, we cross the curve $\mathrm{x}_+^{(Q)}$ that is mapped to the curve $|x_1^-| = 1$ on the z-torus. The cut structure in this region appears to be the same as for the fundamental particle case, and

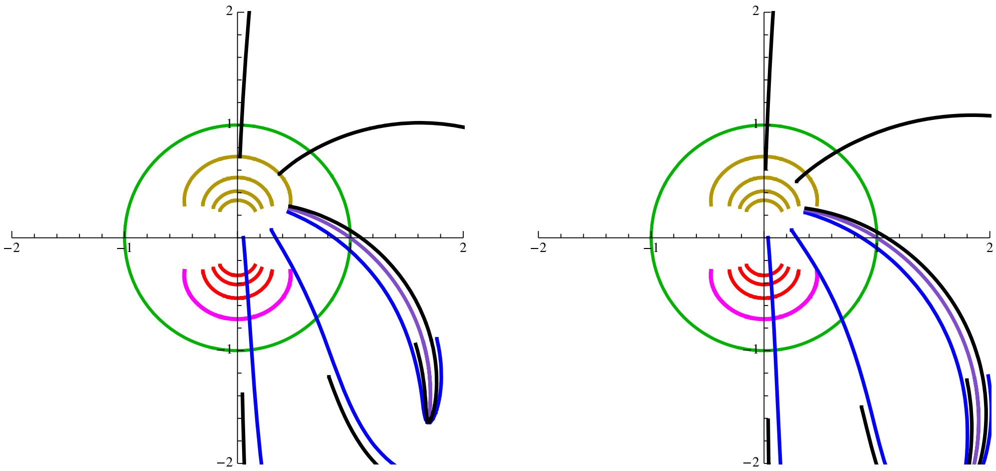


Figure 10. Blue and black curves represent $x^-(z)$ corresponding to the curves $x^+(z)$ in figure 6. No black curve intersects the cuts inside the circle but they all touch the upper cut. The blue curves close enough to $\mathrm{Re}(z) = \frac{\omega_1}{4}$ do not intersect the cuts either.

we get the following expressions for the functions χ with the first particle being in the region $\mathcal{R}_{2,0}$:

$$\mathcal{R}_{2,0}: \quad \chi\left(x_1^+, x_2^\pm\right) = \Phi\left(x_1^+, x_2^\pm\right) - \Psi\left(x_1^+, x_2^\pm\right) - \frac{1}{\mathrm{i}} \log \prod_{j=1}^{Q-1} \frac{w_j^-\left(x_1^+\right) - x_2^\pm}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^\pm} \tag{5.8}$$

$$+ \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^\pm}{x_1^- - x_2^\pm},$$

$$\chi\left(x_1^-, x_2^\pm\right) = \Phi\left(x_1^-, x_2^\pm\right) - \Psi\left(x_1^-, x_2^\pm\right).$$

It is worth noting that the cuts of the functions $w_n^-(x_1^+)$ on the z-torus coincide with the images of the curve $\mathrm{x}_+^{(n)}$, and therefore, they are already included in the cut structure of the χ-functions. As we will show in the next subsection, the dressing factor satisfies the crossing equation and is an analytic function in $\mathcal{R}_{2,0}$.

Region $\mathcal{R}_{1,1}$*:* $\{z_1, z_2\} \in \mathcal{R}_{1,1} \implies |x_1^+| < 1,\ |x_1^-| > 1;\ |x_2^+| < 1,\ |x_2^-| > 1$
Finding the analytic continuation to the region $\mathcal{R}_{1,1}$ basically follows the ones in the previous section, and for $\mathcal{R}_{1,0}$. Since the second particle is an M-particle bound state, the region $|x_2^+| < 1$, $|x_2^-| > 1$ should be divided into M smaller regions and, as a consequence, $\mathcal{R}_{1,1}$ should be understood as a union of $Q \times M$ regions which we denote as $\mathcal{R}_{1,1}^{n,m}$, $n = 1, \ldots, Q$, $m = 1, \ldots, M$. Then, we begin with formulae (5.1) for the functions χ in the region $\mathcal{R}_{1,0}$, and analytically continue in z_2. The resulting expressions for the functions χ appear as follows:

$$\mathcal{R}_{1,1}^{n,m}: \quad \chi\left(x_1^+, x_2^+\right) = \Phi\left(x_1^+, x_2^+\right) + \Psi\left(x_2^+, x_1^+\right) - \Psi\left(x_1^+, x_2^+\right) - \frac{1}{\mathrm{i}} \log \prod_{j=1}^{n-1} \frac{w_j^-\left(x_1^+\right) - x_2^+}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^+}$$

$$+ \frac{1}{\mathrm{i}} \log \prod_{j=1}^{m-1} \frac{w_j^-\left(x_2^+\right) - x_1^+}{\frac{1}{w_j^-\left(x_2^+\right)} - x_1^+} + \mathrm{i} \log \frac{\Gamma\left[1 + \frac{\mathrm{i}}{2} g \left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2} g \left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]},$$

$$
\begin{aligned}
&\chi\left(x_1^+, x_2^-\right) = \Phi\left(x_1^+, x_2^-\right) - \Psi\left(x_1^+, x_2^-\right) - \frac{1}{\mathrm{i}} \log \prod_{j=1}^{n-1} \frac{w_j^-\left(x_1^+\right) - x_2^-}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^-}, \\
&\chi\left(x_1^-, x_2^+\right) = \Phi\left(x_1^-, x_2^+\right) + \Psi\left(x_2^+, x_1^-\right) + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{m-1} \frac{w_j^-\left(x_2^+\right) - x_1^-}{\frac{1}{w_j^-\left(x_2^+\right)} - x_1^-}, \\
&\chi\left(x_1^-, x_2^-\right) = \Phi\left(x_1^-, x_2^-\right).
\end{aligned} \tag{5.9}
$$

Region $\mathcal{R}_{2,1}$*:* $\{z_1, z_2\} \in \mathcal{R}_{2,1} \implies |x_1^+| < 1,\ |x_1^-| < 1;\ |x_2^+| < 1,\ |x_2^-| > 1$
Shifting z_1 further upward into the anti-particle region, we get in the region $\mathcal{R}_{2,1}^m, m = 1, \ldots, M$ bounded by the curves $\mathcal{X}_+^{(m-1)}$ and $\mathcal{X}_+^{(m)}$ in the z_2-torus. The corresponding analytic continuation reads as:

$$
\begin{aligned}
\mathcal{R}_{2,1}^m: \quad &\chi\left(x_1^+, x_2^+\right) = \Phi\left(x_1^+, x_2^+\right) + \Psi\left(x_2^+, x_1^+\right) - \Psi\left(x_1^+, x_2^+\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^+}{x_1^- - x_2^+} \\
&\quad - \frac{1}{\mathrm{i}} \log \prod_{j=1}^{Q-1} \frac{w_j^-\left(x_1^+\right) - x_2^+}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^+} + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{m-1} \frac{w_j^-\left(x_2^+\right) - x_1^+}{\frac{1}{w_j^-\left(x_2^+\right)} - x_1^+} \\
&\quad + \mathrm{i} \log \frac{\Gamma\left[1 + \frac{\mathrm{i}}{2} g \left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2} g \left(x_1^+ + \frac{1}{x_1^+} - x_2^+ - \frac{1}{x_2^+}\right)\right]}, \\
&\chi\left(x_1^+, x_2^-\right) = \Phi\left(x_1^+, x_2^-\right) - \Psi\left(x_1^+, x_2^-\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-} - \frac{1}{\mathrm{i}} \log \prod_{j=1}^{Q-1} \frac{w_j^-(x_1^+) - x_2^-}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^-}, \\
&\chi\left(x_1^-, x_2^+\right) = \Phi\left(x_1^-, x_2^+\right) - \Psi\left(x_1^-, x_2^+\right) + \Psi\left(x_2^+, x_1^-\right) + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{m-1} \frac{w_j^-\left(x_2^+\right) - x_1^-}{\frac{1}{w_j^-\left(x_2^+\right)} - x_1^-} \\
&\quad + \mathrm{i} \log \frac{\Gamma\left[1 + \frac{\mathrm{i}}{2} g \left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2} g \left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}, \\
&\chi\left(x_1^-, x_2^-\right) = \Phi\left(x_1^-, x_2^-\right) - \Psi\left(x_1^-, x_2^-\right).
\end{aligned} \tag{5.10}
$$

Region $\mathcal{R}_{3,1}$*:* $\{z_1, z_2\} \in \mathcal{R}_{3,1} \implies |x_1^+| > 1,\ |x_1^-| < 1;\ |x_2^+| < 1,\ |x_2^-| > 1$
Moving further upward into the region $|x_1^-| < 1, |x_1^+| > 1$, the point z_1 crosses the curve $x_-^{(Q)}$ that is mapped to the curve $|x_1^+| = 1$ on the z-torus and is the upper boundary of the region $\mathcal{R}_{2,1}$. Then, the proper analytic continuation requires us to cross the images of the curves $\mathrm{x}_-^{(n)}$, $n = Q - 1, \ldots, 1$, and that means that the corresponding cuts on the z-torus should be the images of the curves $\check{x}_-^{(n)} = \frac{1}{x(u - \frac{2\mathrm{i}}{g} n)}$, $n = Q - 1, \ldots, 1$, where the parameter u again takes values in $|u| \geqslant 2$. It is not difficult to see that for $n = 1, \ldots, Q - 1$ the images of $\mathrm{x}_-^{(n)}$ and $\check{x}_-^{(n)}$ in the region $\mathcal{R}_{3,1}$ can be obtain from those for $\mathrm{x}_+^{(Q-n)}$ and $\check{x}_+^{(Q-n)}$ in the region $\mathcal{R}_{1,1}$ just by shifting them upward by ω_2.

Thus, the region $\mathcal{R}_{3,1}$ is also divided by the curves $\mathcal{X}_-^{(n)}$ which are the union of the curves $x_-^{(Q-n)}$ and $\check{x}_-^{(Q-n)}$ into Q smaller regions $\mathcal{R}_{3,1}^n$ bounded by the curves $\mathcal{X}_-^{(n-1)}$ and $\mathcal{X}_-^{(n)}$, where $n = 1, \ldots, Q$. To reach the region $\mathcal{R}_{3,1}^n$, one should analytically continue across the $n - 1$

curves $\mathrm{x}_-^{(Q-1)}, \ldots, \mathrm{x}_-^{(Q-n+1)}$. A new subtlety of the bound state case is that once x_1^- in the x-plane crosses $\mathrm{x}_-^{(Q-n)}$, $n = 1, \ldots, Q-1$, the parameter x_1^+ with $|x_1^+| > 1$ crosses the curves $1/\mathrm{x}_-^{(n)}$ which are outside the unit circle. These curves are the cuts of the functions $w_n^-(x_1^+)$, and therefore, one should replace them with $1/w_n^-(x_1^+)$ after having crossed the cuts.

Repeating the consideration in the previous section and taking into account the new effects, one then gets that the analytic continuation is given by

$$
\begin{aligned}
\mathcal{R}_{3,1}^{n,m}: \chi\left(x_1^+, x_2^+\right) &= \Phi\left(x_1^+, x_2^+\right) + \Psi\left(x_2^+, x_1^+\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^+}{x_1^- - x_2^+} \\
&\quad - \frac{1}{\mathrm{i}} \log \prod_{j=n}^{Q-1} \frac{w_j^-\left(x_1^+\right) - x_2^+}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^+} + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{m-1} \frac{w_j^-\left(x_2^+\right) - x_1^+}{\frac{1}{w_j^-\left(x_2^+\right)} - x_1^+} + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{n-1} \frac{w_j^-\left(x_1^+\right) - x_2^+}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^+}, \\
\chi\left(x_1^+, x_2^-\right) &= \Phi\left(x_1^+, x_2^-\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-} - \frac{1}{\mathrm{i}} \log \prod_{j=n}^{Q-1} \frac{w_j^-\left(x_1^+\right) - x_2^-}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^-} \\
&\quad + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{n-1} \frac{w_j^-\left(x_1^+\right) - x_2^-}{\frac{1}{w_j^-\left(x_1^+\right)} - x_2^-}, \\
\chi\left(x_1^-, x_2^+\right) &= \Phi\left(x_1^-, x_2^+\right) - \Psi\left(x_1^-, x_2^+\right) + \Psi\left(x_2^+, x_1^-\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^+} - x_2^+}{x_1^+ - x_2^+} \\
&\quad + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{n-1} \frac{w_{Q-j}^+\left(x_1^-\right) - x_2^+}{\frac{1}{w_{Q-j}^+\left(x_1^-\right)} - x_2^+} + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{m-1} \frac{w_j^-\left(x_2^+\right) - x_1^-}{\frac{1}{w_j^-\left(x_2^+\right)} - x_1^-} \\
&\quad + \mathrm{i} \log \frac{\Gamma\left[1 + \frac{\mathrm{i}}{2} g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2} g\left(x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}\right)\right]}, \\
\chi\left(x_1^-, x_2^-\right) &= \Phi\left(x_1^-, x_2^-\right) - \Psi\left(x_1^-, x_2^-\right) + \frac{1}{\mathrm{i}} \log \frac{\frac{1}{x_1^+} - x_2^-}{x_1^+ - x_2^-} \\
&\quad + \frac{1}{\mathrm{i}} \log \prod_{j=1}^{n-1} \frac{w_{Q-j}^+\left(x_1^-\right) - x_2^-}{\frac{1}{w_{Q-j}^+\left(x_1^-\right)} - x_2^-}.
\end{aligned}
\tag{5.11}
$$

The cut structure in the region $\mathcal{R}_{3,1}$ is the same as that in $\mathcal{R}_{1,1}$. The analytic continuation of the functions χ is quite complicated. On the other hand, as we will show in the next subsection, the dressing phase satisfies the crossing equation, and therefore differs from the one in the region $\mathcal{R}_{1,1}$ just by a simple crossing equation term.

5.2. The crossing equations for bound states

The crossing equations for the dressing factors involving Q-particle and M-particle bound states are given in (2.14), and the $x^{\pm}$ variables of the bound states satisfy the constraints (2.13). In this section, we consider only the crossing equation with respect to the first argument shifted upward from the particle region $\mathcal{R}_{0,0}$ and from the region $\mathcal{R}_{1,1}$.

We start with the particle region $\mathcal{R}_{0,0}$. By using equation (5.8) and the identity (3.4), we get

$$\Delta\theta = \Psi\left(\frac{1}{x_1^-}, x_2^+\right) - \Psi\left(\frac{1}{x_1^+}, x_2^+\right) + \Psi\left(\frac{1}{x_1^+}, x_2^-\right) - \Psi\left(\frac{1}{x_1^-}, x_2^-\right)$$
$$+ \frac{1}{\mathrm{i}}\log\frac{x_1^- - x_2^+}{\frac{1}{x_1^-} - x_2^+}\frac{\frac{1}{x_1^-} - x_2^-}{x_1^- - x_2^-} - \frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q-1}\frac{w_j^-(x_1^+) - x_2^+}{\frac{1}{w_j^-(x_1^+)} - x_2^+}\frac{\frac{1}{w_j^-(x_1^+)} - x_2^-}{w_j^-(x_1^+) - x_2^-}. \tag{5.12}$$

Taking into account that for $|x_1^\pm| > 1$ and $|x_2| > 1$,

$$\Psi\left(\frac{1}{x_1^-}, x_2\right) - \Psi\left(\frac{1}{x_1^+}, x_2\right) = \frac{1}{\mathrm{i}}\oint\frac{\mathrm{d}w}{2\pi\,\mathrm{i}}\log(w - x_2)\left(w - \frac{1}{w}\right) \tag{5.13}$$
$$\times\sum_{j=1}^{Q}\left[\frac{1}{(w - w_j^+(x_1^-))\left(w - \frac{1}{w_j^+(x_1^-)}\right)} + \frac{1}{(w - w_j^-(x_1^+))\left(w - \frac{1}{w_j^-(x_1^+)}\right)}\right]$$
$$= -\frac{2Q}{\mathrm{i}}\log x_2 + \frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q}\left(w_j^+(x_1^-) - x_2\right)\left(w_j^-(x_1^+) - x_2\right),$$

we obtain

$$\Psi\left(\frac{1}{x_1^-}, x_2^+\right) - \Psi\left(\frac{1}{x_1^+}, x_2^+\right) + \Psi\left(\frac{1}{x_1^+}, x_2^-\right) - \Psi\left(\frac{1}{x_1^-}, x_2^-\right) = \tag{5.14}$$
$$= \frac{2Q}{\mathrm{i}}\log\frac{x_2^-}{x_2^+} + \frac{1}{\mathrm{i}}\log\frac{\frac{1}{x_1^+} - x_2^+}{\frac{1}{x_1^+} - x_2^-}\frac{\frac{1}{x_1^-} - x_2^+}{\frac{1}{x_1^-} - x_2^-} + \frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q-1}\frac{\left(w_j^+(x_1^-) - x_2^+\right)\left(w_j^-(x_1^+) - x_2^+\right)}{\left(w_j^+(x_1^-) - x_2^-\right)\left(w_j^-(x_1^+) - x_2^-\right)},$$

where we singled out the contribution with $j = Q$. By using equation (5.14), we find that equation (5.12) takes the form

$$\Delta\theta = \left(\frac{x_2^-}{x_2^+}\right)^Q\frac{x_1^- - x_2^+}{x_1^- - x_2^-}\frac{1 - \frac{1}{x_1^+x_2^+}}{1 - \frac{1}{x_1^+x_2^-}}\prod_{j=1}^{Q-1}\frac{\left(w_j^+(x_1^-) - x_2^+\right)\left(1 - \frac{1}{w_j^-(x_1^+)x_2^+}\right)}{\left(w_j^+(x_1^-) - x_2^-\right)\left(1 - \frac{1}{w_j^-(x_1^+)x_2^-}\right)}.$$

Finally, taking into account that $w_j^-(x_1^+) = w_{Q-j}^+(x_1^-)$, we obtain

$$\Delta\theta = \left(\frac{x_2^-}{x_2^+}\right)^Q\frac{x_1^- - x_2^+}{x_1^- - x_2^-}\frac{1 - \frac{1}{x_1^+x_2^+}}{1 - \frac{1}{x_1^+x_2^-}}\prod_{j=1}^{Q-1}G(M - Q + 2j),$$

that is the correct crossing equation (2.14).

To discuss the crossing equation for the region $\mathcal{R}_{1,1}^{n,m}$, it is convenient to split the contribution of the sum of phases into the following four parts:

$$\Delta\theta = \Delta_1\theta + \Delta_2\theta + \Delta_3\theta + \Delta_4\theta, \tag{5.15}$$

where $\Delta_1\theta$ is the contribution of the Ψ-functions

$$\Delta_1\theta = \Psi\left(\frac{1}{x_1^-}, x_2^+\right) - \Psi\left(\frac{1}{x_1^+}, x_2^+\right) + \Psi\left(\frac{1}{x_1^+}, x_2^-\right) - \Psi\left(\frac{1}{x_1^-}, x_2^-\right), \tag{5.16}$$

$\Delta_2\theta$ is the contribution of the Γ-functions

$$\Delta_2\theta = \frac{1}{\mathrm{i}}\log\frac{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(x_1^-+\frac{1}{x_1^-}-x_2^+-\frac{1}{x_2^+}\right)\right]}{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(x_1^-+\frac{1}{x_1^-}-x_2^+-\frac{1}{x_2^+}\right)\right]}\frac{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(x_1^++\frac{1}{x_1^+}-x_2^+-\frac{1}{x_2^+}\right)\right]}{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(x_1^++\frac{1}{x_1^+}-x_2^+-\frac{1}{x_2^+}\right)\right]}, \tag{5.17}$$

$\Delta_3\theta$ is the contribution due to crossing the upper boundaries of the regions $\mathcal{R}_{0,0}$, $\mathcal{R}_{1,0}$, $\mathcal{R}_{1,1}$ and $\mathcal{R}_{2,1}$

$$\Delta_3\theta = \frac{1}{\mathrm{i}}\log\frac{x_1^- - x_2^+}{\frac{1}{x_1^-}-x_2^+}\frac{\frac{1}{x_1^-}-x_2^-}{x_1^- - x_2^-}\frac{x_1^+ - x_2^-}{\frac{1}{x_1^+}-x_2^-}\frac{\frac{1}{x_1^+}-x_2^+}{x_1^+ - x_2^+} \tag{5.18}$$

and $\Delta_4\theta$ is the contribution due to crossing the curves $\mathrm{x}_\pm^{(1)}, \ldots, \mathrm{x}_\pm^{(Q-1)}$

$$\Delta_4\theta = -\frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q-1}\frac{w_j^-(x_1^+)-x_2^+}{\frac{1}{w_j^-(x_1^+)}-x_2^+}\frac{\frac{1}{w_j^-(x_1^+)}-x_2^-}{w_j^-(x_1^+)-x_2^-}. \tag{5.19}$$

We see that any dependence on n and m disappears. To prove the crossing equation, we first generalize the identity (A.9)

$$\Psi\left(\frac{1}{x_1^-},0\right)-\Psi\left(\frac{1}{x_1^+},0\right) = \mathrm{i}Q\log\frac{g^2}{4}+\frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q}w_j^-(x_1^+)w_j^+(x_1^-). \tag{5.20}$$

Next, we take into account that in this region $|x^+|<1$, $|x^-|>1$ and therefore, $w_Q^-(x_1^+)=\frac{1}{x_1^-}$, $w_Q^+(x_1^-)=x_2^+$. By using equations (5.13), (3.15) and (5.20), we find

$$\begin{aligned}\Delta_1\theta = {}& \frac{Q}{\mathrm{i}}\log\frac{4}{g^2}+\frac{2Q-1}{\mathrm{i}}\log\frac{x_2^-}{x_2^+}-\frac{1}{\mathrm{i}}\log\left(x_1^+-x_2^-\right)\left(x_1^- - x_2^+\right)\left(1-\frac{1}{x_1^+x_2^+}\right)\left(1-\frac{1}{x_1^-x_2^-}\right)\\ &-\frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q-1}\left(w_j^-(x_1^+)-x_2^-\right)\left(1-\frac{1}{w_j^-(x_1^+)x_2^+}\right)\left(w_j^+(x_1^-)-x_2^-\right)\left(1-\frac{1}{w_j^+(x_1^-)x_2^+}\right).\end{aligned} \tag{5.21}$$

Simplifying $\Delta_2\theta$, we get

$$\begin{aligned}\Delta_2\theta = {}& \frac{Q}{\mathrm{i}}\log\frac{g^2}{4}+\frac{1}{\mathrm{i}}\log\left(x_1^+-x_2^+\right)\left(x_1^- - x_2^+\right)\left(1-\frac{1}{x_1^-x_2^+}\right)\left(1-\frac{1}{x_1^+x_2^+}\right)\\ &+\frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q-1}\left(w_j^-(x_1^+)-x_2^+\right)\left(1-\frac{1}{w_j^-(x_1^+)x_2^+}\right)\left(w_j^+(x_1^-)-x_2^+\right)\left(1-\frac{1}{w_j^+(x_1^-)x_2^+}\right).\end{aligned} \tag{5.22}$$

Summing up all the contributions, we again obtain the crossing equation (2.14).

6. Bound state dressing factor of mirror theory

In this section, we determine the dressing factor which encodes scattering of bound states in the mirror theory and analyze some of its properties. Recall that Q-particle bound states of the mirror theory obey the following equations:

$$x_1^- = x_2^+\,,\quad x_2^- = x_3^+\,,\quad \ldots,\quad x_{Q-1}^- = x_Q^+\,. \tag{6.1}$$

These equations have 2^{Q-1} different solutions sharing the same set of conserved charges. One of these solutions has all the constituent particles located in the region $\mathrm{Im}(x_j^\pm) < 0$, which has been identified in [9] as the mirror region. In general, the constituent particles lie anywhere on the z-torus and therefore, their individual dressing factors should be determined by using the analytic continuation we have established in section 4.

As in the string theory case, the mirror bound state dressing factor $\sigma^{QQ'}$ can be found by fusing the dressing factors of the constituent particles. However, in dealing with TBA equations, we are primarily interested not in the dressing factor itself but rather in the following quantity:

$$\Sigma^{QQ'} = \sigma^{QQ'} \prod_{j=1}^{Q}\prod_{k=1}^{Q'} \frac{1-\frac{1}{x_j^+ z_k^-}}{1-\frac{1}{x_j^- z_k^+}}, \tag{6.2}$$

because its logarithmic derivative appears as one of the TBA kernels [30]. Here $x_j^\pm$ and $z_k^\pm$ are the kinematical parameters of the constituent particles corresponding to Q- and Q'-particle bound states, respectively, and

$$\sigma^{QQ'} = \prod_{j=1}^{Q}\prod_{k=1}^{Q'} \sigma(x_j, z_k). \tag{6.3}$$

The quantity $\Sigma^{QQ'}$ arises from fusion of the scalar factors of mirror theory scattering matrices corresponding to the constituent particles. Since the bound state equations have many solutions, it is *a priori* unclear which one should be used in the fusion procedure. Also, in contrast to the string theory bound state S-matrix, the product factor on the right-hand side of equation (6.2) does depend on the internal structure of the bound states involved. Indeed, defining the bound state kinematic parameters as

$$y_1^+ = x_1^+, \quad y_1^- = x_Q^-, \quad y_2^+ = z_1^+, \quad y_2^- = z_{Q'}^- \tag{6.4}$$

and using the bound state equations, the product factor can be represented as

$$\prod_{k=1}^{Q'}\prod_{j=1}^{Q} \frac{1-\frac{1}{x_j^+ z_k^-}}{1-\frac{1}{x_j^- z_k^+}} = \frac{1-\frac{1}{y_1^+ y_2^-}}{1-\frac{1}{y_1^- y_2^+}} \prod_{j=1}^{Q-1} \frac{1-\frac{1}{x_j^- y_2^-}}{1-\frac{1}{x_j^- y_2^+}} \prod_{k=1}^{Q'-1} \frac{1-\frac{1}{y_1^+ z_k^-}}{1-\frac{1}{y_1^- z_k^-}}, \tag{6.5}$$

which makes this dependence manifest. On the other hand, in the physical mirror theory we might expect to find a unique bound state scattering matrix. Indeed, as we will argue, the choice of a bound state solution is just a matter of convenience and all 2^{Q-1} solutions lead to one and the same scalar factor $\Sigma^{QQ'}$ and, as a result, to the same bound state S-matrix.

Before studying the dressing factor in full generality, we consider Σ^{Q1} and evaluate it on a particular bound state solution. The most convenient choice is provided by the bound state solution used in [45]. Indeed, for this solution only, the first particle occurs in the region $\mathcal{R}_1$, while all the others fall in the particle region:

$$\left|x_j^-\right| > 1\,,\ \left|x_{j+1}^+\right| > 1\,,\ j = 1,\dots,Q\,,\qquad \left|x_1^+\right| < 1\,,\ \mathrm{Im}\left(x_1^+\right) < 0\,,\ \mathrm{Im}\left(x_Q^-\right) < 0 \tag{6.6}$$

and therefore, this solution requires a minimal amount of analytic continuation. In terms of the function $x(u)$ given by (3.9), this solution reads as

$$x_j^- = x\left(u + \frac{\mathrm{i}}{g}(Q-2j)\right), \quad j = 1,\dots,Q\,, \tag{6.7}$$

$$x_1^+ = \frac{1}{x\left(u+\frac{\mathrm{i}}{g}Q\right)}, \quad x_j^+ = x\left(u + \frac{\mathrm{i}}{g}(Q-2j+2)\right), \quad j = 2,\dots,Q\,.$$

For $Q = 2m$, the middle particles with $j = m, m+1$ can be on the cut of $x(u)$.

As a warmup exercise, we will first consider the case where the second particle is a fundamental particle located in the particle region. The dressing phase is obtained through the fusion procedure

$$\theta(y_1, y_2) = \sum_{j=1}^{Q} \theta(x_j, y_2), \quad y_1^+ = x_1^+, \quad y_1^- = x_Q^-, \tag{6.8}$$

where $|y_2^\pm| > 1$ since these parameters correspond to the second particle. A simple computation making use of solution (6.7) gives

$$\frac{1}{\mathrm{i}} \log \Sigma^{Q1}(y_1, y_2) = \Phi(y_1^+, y_2^+) - \Psi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-) + \Psi(y_1^+, y_2^-) \tag{6.9}$$
$$- \Phi(y_1^-, y_2^+) + \Phi(y_1^-, y_2^-) + \frac{1}{\mathrm{i}} \log \frac{1 - \frac{1}{y_1^+ y_2^-}}{1 - \frac{1}{y_1^- y_2^+}} \prod_{j=1}^{Q-1} \frac{1 - \frac{1}{x\left(u + \frac{1}{g}(Q-2j)\right) y_2^-}}{1 - \frac{1}{x\left(u + \frac{1}{g}(Q-2j)\right) y_2^+}}.$$

Here we have also applied formulas (4.1) and (4.2) to account for the dressing phase of the first particle being in the region $\mathcal{R}_1$; the logarithmic factor on the right-hand side comes from equation (6.5).

The logarithmic term in the expression (6.9) exhibits an explicit dependence on the kinematic parameters of the constituent particles. As we will now see, this dependence is, however, artificial and can be completely removed by making appropriate transformations of Ψ-functions. First, as is shown in Appendix 19, the following formula is valid:

$$\Psi(y_1^+, y_2^-) - \Psi(y_1^-, y_2^-) = -\frac{1}{\mathrm{i}} \log \left(\frac{y_1^+}{y_2^-} - 1\right) \left(\frac{1}{y_1^- y_2^-} - 1\right) \prod_{j=1}^{Q-1} \left(1 - \frac{1}{x_j^- y_2^-}\right)^2. \tag{6.10}$$

Second, we rewrite equation (6.9) in the following form:

$$\frac{1}{\mathrm{i}} \log \Sigma^{Q1}(y_1, y_2) = \Phi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-) - \Phi(y_1^-, y_2^+) + \Phi(y_1^-, y_2^-) \tag{6.11}$$
$$+ \frac{1}{2} \left[-\Psi(y_1^+, y_2^+) - \underbrace{\Psi(y_1^+, y_2^+)} + \Psi(y_1^+, y_2^-) + \underbrace{\Psi(y_1^+, y_2^-)} \right]$$
$$+ \frac{1}{\mathrm{i}} \log \frac{1 - \frac{1}{y_1^+ y_2^-}}{1 - \frac{1}{y_1^- y_2^+}} \prod_{j=1}^{Q-1} \frac{1 - \frac{1}{x\left(u + \frac{1}{g}(Q-2j)\right) y_2^-}}{1 - \frac{1}{x\left(u + \frac{1}{g}(Q-2j)\right) y_2^+}}.$$

Finally, by using formula (6.10), we substitute the first and the second underbraced Ψ-functions for $\Psi(y_1^-, y_2^+)$ and $\Psi(y_1^-, y_2^-)$, respectively, and obtain the following neat result:

$$\frac{1}{\mathrm{i}} \log \Sigma^{Q1}(y_1, y_2) = \Phi(y_1^+, y_2^+) - \Phi(y_1^+, y_2^-) - \Phi(y_1^-, y_2^+) + \Phi(y_1^-, y_2^-) \tag{6.12}$$
$$+ \frac{1}{2} \left[-\Psi(y_1^+, y_2^+) - \Psi(y_1^-, y_2^+) + \Psi(y_1^+, y_2^-) + \Psi(y_1^-, y_2^-) \right]$$
$$+ \frac{1}{2\mathrm{i}} \log \frac{(y_1^+ - y_2^+) \left(y_2^- - \frac{1}{y_1^+}\right)^2}{(y_1^+ - y_2^-) \left(y_2^- - \frac{1}{y_1^-}\right) \left(y_2^+ - \frac{1}{y_1^-}\right)}.$$

Quite fascinating, all the dependence on the constituent particles has completely cancelled out. Although, to obtain the dressing factor we started from a particular bound state solution, the final result does not bear any reminiscence of this particularity. This provides a strong indication that the same universal answer will be obtained by starting from any of 2^{Q-1} solutions. In fact, in Appendix 18, we provide another derivation of the corresponding dressing factor starting from the bound state solution with all constituent particles being in the mirror region $\mathrm{Im}\, x^{\pm} < 0$ and show that it leads to the same answer as above.

Now we are ready to obtain the general bound state factor $\Sigma^{QQ'}$ of the mirror theory. Our success in showing the decoupling of bound state constituents from the final answer motivates us to start again from the solution (6.7) for both Q and Q' bound states. Applying a similar reasoning as before, we find

$$\begin{aligned}\frac{1}{\mathrm{i}}\log\Sigma^{QQ'}(y_1,y_2) &= \Phi(y_1^+,y_2^+)-\Phi(y_1^+,y_2^-)-\Phi(y_1^-,y_2^+)+\Phi(y_1^-,y_2^-)\\ &\quad-\Psi(y_1^+,y_2^+)+\Psi(y_1^+,y_2^-)+\Psi(y_2^+,y_1^+)-\Psi(y_2^+,y_1^-)\\ &\quad+\frac{1}{\mathrm{i}}\log\frac{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}\\ &\quad+\frac{1}{\mathrm{i}}\log\frac{1-\frac{1}{y_1^+y_2^-}}{1-\frac{1}{y_1^-y_2^+}}\prod_{j=1}^{Q-1}\frac{1-\frac{1}{x_j^-y_2^-}}{1-\frac{1}{x_j^-y_2^+}}\prod_{k=1}^{Q'-1}\frac{1-\frac{1}{y_1^+z_k^-}}{1-\frac{1}{y_1^-z_k^-}}.\end{aligned}\tag{6.13}$$

This formula was derived by summing up the corresponding contributions coming from four regions $\mathcal{R}_{1,1}$, $\mathcal{R}_{1,0}$, $\mathcal{R}_{0,1}$ and $\mathcal{R}_{0,0}$. The logarithmic factor in the last line comes from equation (6.5) and, as before, it contains the explicit dependence on the bound state constituents.

To get rid of the bound state constituents in equation (6.13), we can try a similar trick as in the previous case. By using formula (A.33) worked out in Appendix 19, we find

$$\begin{aligned}\frac{1}{\mathrm{i}}\log\Sigma^{QQ'}(y_1,y_2) &= \Phi(y_1^+,y_2^+)-\Phi(y_1^+,y_2^-)-\Phi(y_1^-,y_2^+)+\Phi(y_1^-,y_2^-)\\ &\quad-\frac{1}{2}\left(\Psi(y_1^+,y_2^+)+\Psi(y_1^-,y_2^+)-\Psi(y_1^+,y_2^-)-\Psi(y_1^-,y_2^-)\right)\\ &\quad+\frac{1}{2}\left(\Psi(y_2^+,y_1^+)+\Psi(y_2^-,y_1^+)-\Psi(y_2^+,y_1^-)-\Psi(y_2^-,y_1^-)\right)\\ &\quad+\frac{1}{\mathrm{i}}\log\frac{\mathrm{i}^Q\,\Gamma\left[Q'-\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}{i^{Q'}\Gamma\left[Q+\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}\,\frac{1-\frac{1}{y_1^+y_2^-}}{1-\frac{1}{y_1^-y_2^+}}\sqrt{\frac{y_1^+y_2^-}{y_1^-y_2^+}}\,.\end{aligned}\tag{6.14}$$

This formula represents the (logarithm of) improved bound state dressing factor $\Sigma^{QQ'}$ of the mirror theory. It is manifestly antisymmetric under interchanging the particles $1\leftrightarrow 2$. Since for physical particles of the mirror theory conjugation acts as $(y^{\pm})^* = 1/y^{\mp}$, one can also see that the dressing factor appears to be unitary. Indeed, the last term in equation (6.14) is real, while to prove the reality of the remaining terms one has to use identities (3.4) and (3.15). Most remarkably, the factor (6.14) depends on the kinematic variables $y_i^{\pm}$ only, which points to the uniqueness of the mirror S-matrix and, therefore, to the validity of the whole mirror approach. This factor also provides the final missing piece in the derivation of the TBA equations for the $\mathrm{AdS}_5\times\mathrm{S}^5$ mirror model [30].

Concerning the analytic properties of $\Sigma^{QQ'}$, the cuts $\check{x}_+^{(n)}$ in the region $\mathcal{R}_1$ are cancelled[10] in the sum $\Psi(y_2^+,y_1^+)+\Psi(y_2^-,y_1^+)$. Analogous cancellation takes place in the other sums in equation (6.14). As a result, the improved dressing factor $\Sigma^{QQ'}$ is a holomorphic function in the intersection of the regions $\mathcal{R}_{1,1}$ and $\mathrm{Im}\, y_i^{\pm} < 0$. We also note that $\Sigma^{QQ'}$ being continued

[10] The same is true for the cuts $\check{x}_-^{(n)}$ in the region $\mathcal{R}_1$, as one could expect from the validity of the crossing equations.

in the particle region will have there the cuts which are analogous to those of the bound state dressing factor of string theory in the region $\mathcal{R}_{1,1}$.

Finally, we mention that we have also made a computation of the full mirror theory factor $\Sigma^{QQ'}$ by using the bound state solution (A.13) and verified that it leads to the same result as the solution (6.7).

7. Conclusions

In this paper we analyzed the analytic continuation of the BES dressing factor for a string bound state S-matrix on the z-torus, and found its branch where the crossing equations are satisfied. This provides a new test of the BES proposal.

In the particle region, the dressing factor of any bound state S-matrix can be written in a universal form in terms of a single function χ of two variables. Our results however show that the analytic continuation and the choice of the branch of the dressing factor depend on the type of scattered bound states and do not follow just from an analytic continuation of the function χ.

The analytic properties of the dressing factor for fundamental particles appear to be better than one could anticipate. In particular, it is an analytic function in (the square of) the union of the particle region and the region $\mathrm{Im}(x^{\pm}) < 0$, and regions obtained from this union by shifting it upward or downward by ω_2. The region $\mathrm{Im}(x^{\pm}) < 0$ was considered in [9] as a candidate for the physical region of the mirror model, and since the dressing factor is analytic in this region, there might exist an integral representation for the factor, similar to the DHM one, which makes analyticity manifest.

The string theory bound state dressing factor, however, is not analytic in the region $\mathrm{Im}(x^{\pm}) < 0$ and has there a finite number of cuts equal to $Q-1$ for a Q-particle bound state. If Q is even then one of the cuts is located on the real momentum line of the mirror theory. We stress, however, that this discussion concerns the dressing factor for bound states of string theory, but not the one for bound states of the associated mirror model.

We have also determined the bound state dressing factor of the mirror model. It is obtained by fusing the dressing factors of mirror constituent particles that are located in various kinematic regions of the z-torus and, for this reason, do not admit a universal representation for their dressing factor in terms of χ-functions. Breakdown of this universality for particles outside the particle region implies that the resulting bound state dressing factor may depend on the choice of a bound state solution. A Q-particle bound state equation has 2^{Q-1} different solutions, any of them can be used for constructing the corresponding dressing factor leading, therefore, to *a priori* different results. As we have shown, however, the factor $\Sigma^{QQ'}$, which enters the TBA equations, does not suffer from this ambiguity. The choice of a bound state solution is just a matter of convenience and all 2^{Q-1} solutions lead to one and the same physical S-matrix of the mirror theory.

Finally, we mention that the locations of all the cuts of the dressing factor depend only on a single variable z_1 or z_2 preserving the direct product structure of its domain, and this hints at the existence of a uniformizing variable such that the dressing factor considered as a function of these two variables becomes meromorphic.

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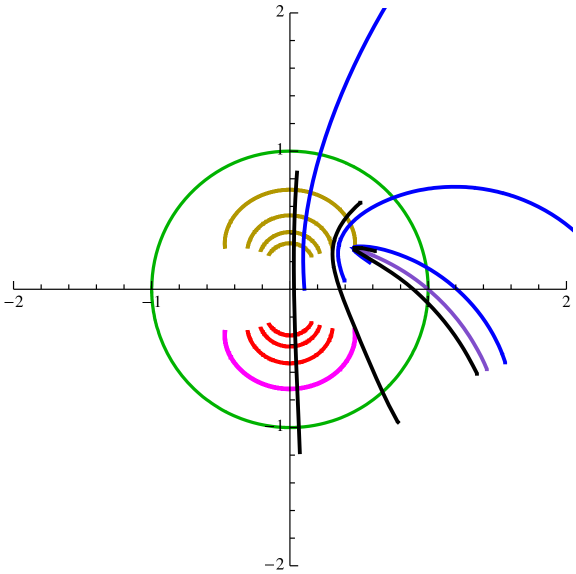

Figure A1. Blue and black curves represent the curves $x^-(z)$ going downward from the real line of the z-torus variable to the line with $\mathrm{Im}(z) = -\omega_2/\mathrm{i}$ and have $|\mathrm{Re}(z)| \leqslant \frac{\omega_1}{4}$, and black curves $x^-(z)$ go downward and have $|\mathrm{Re}(z)| \geqslant \frac{\omega_1}{4}$. Any curve intersects the upper curve in the circle and for the blue curves it is the first one they cross.

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Appendix

Appendix A.1. Some results on the dressing phase for fundamental particles

In this appendix we collect the auxiliary results on the analytic continuation of the dressing phase for fundamental particles to some other regions on the product of two infinite strips $-\frac{\omega_1}{2} \leqslant \mathrm{Im}(z) \leqslant \frac{\omega_1}{2}$. These results can be used, in particular, to verify the crossing equation which arises upon shifting z_2 by $-\omega_2$.

Region $\mathcal{R}_{0,-1}$: $\{z_1, z_2\} \in \mathcal{R}_{0,-1} \implies |x_1^\pm| > 1\,;\ |x_2^+| > 1\,,\ |x_2^-| < 1$
Let us now discuss very briefly the analytic continuation of the dressing phase if the point z_2 is shifted downward, see figure A1. The analytic continuation procedure we have already developed basically repeats itself. Essentially, all we need to do is to change $x_1^\pm \to x_2^\mp$, and to account properly for the signs. Then, we get the following expressions for the functions χ:

$$\mathcal{R}_{0,-1}: \quad \chi\left(x_1^\pm, x_2^-\right) = \Phi\left(x_1^\pm, x_2^-\right) + \Psi\left(x_2^-, x_1^\pm\right), \tag{A.1}$$
$$\chi\left(x_1^\pm, x_2^+\right) = \Phi\left(x_1^\pm, x_2^+\right).$$

Region $\mathcal{R}_{0,-2}$*:* $\{z_1, z_2\} \in \mathcal{R}_{0,-2} \implies |x_1^\pm| > 1\,;\ |x_2^+| < 1\,,\ |x_2^-| < 1$
Shifting the point z_2 further downward into the anti-particle region $\mathcal{R}_{0,-2}$, one gets

$$\mathcal{R}_{0,-2}: \quad \chi\left(x_1^\pm, x_2^-\right) = \Phi\left(x_1^\pm, x_2^-\right) + \Psi\left(x_2^-, x_1^\pm\right) + \frac{1}{\mathrm{i}}\log\frac{x_1^\pm - \frac{1}{x_2^+}}{x_1^\pm - x_2^+}\,, \tag{A.2}$$

$$\chi\left(x_1^\pm, x_2^+\right) = \Phi\left(x_1^\pm, x_2^+\right) + \Psi\left(x_2^+, x_1^\pm\right).$$

These formulas are used to check the crossing equation (2.9).

Region $\mathcal{R}_{-1,-1}$*:* $\{z_1, z_2\} \in \mathcal{R}_{-1,-1} \implies |x_1^+| > 1\, |x_1^-| < 1;\ |x_2^+| > 1\,,\ |x_2^-| < 1$
This region can also be considered as the one containing the real momentum line of the mirror model[11], because its symmetry axis is obtained from one of the particle region by shifting the latter downward by the quarter of the imaginary period. To find the continuation of the χ-functions to this region, we can use the expressions for χs in $\mathcal{R}_{0,-1}$. We find

$$\begin{aligned}
\mathcal{R}_{-1,-1}: \quad & \chi\left(x_1^+, x_2^+\right) = \Phi\left(x_1^+, x_2^+\right)\\
& \chi\left(x_1^+, x_2^-\right) = \Phi\left(x_1^+, x_2^-\right) + \Psi\left(x_2^-, x_1^+\right)\\
& \chi\left(x_1^-, x_2^+\right) = \Phi\left(x_1^-, x_2^+\right) - \Psi\left(x_1^-, x_2^+\right),\\
& \chi\left(x_1^-, x_2^-\right) = \Phi\left(x_1^-, x_2^-\right) + \Psi\left(x_2^-, x_1^-\right) - \Psi\left(x_1^-, x_2^-\right)\\
& \qquad + \mathrm{i}\log\frac{\Gamma\left[1 + \frac{\mathrm{i}}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^- - \frac{1}{x_2^-}\right)\right]}{\Gamma\left[1 - \frac{\mathrm{i}}{2}g\left(x_1^- + \frac{1}{x_1^-} - x_2^- - \frac{1}{x_2^-}\right)\right]}\,.
\end{aligned} \tag{A.3}$$

In fact, the dressing factor in this region can be easily obtained from the one in the region $\mathcal{R}_{1,1}$ by using the crossing equations with both arguments shifted by $-\omega_2$. The dressing factors differ just by a simple factor of the form $\frac{x_1^+ x_2^-}{x_1^- x_2^+}$.

Appendix A.2 Identities for the Ψ-function

Here we present a list of identities satisfied by the Ψ-function, which have been used in proving the crossing equations for the dressing phase of both fundamental and mirror particles.

For $|x_1^\pm| > 1$ and $|x_2| > 1$ the following identity is valid:

$$\begin{aligned}
&\Psi\left(\frac{1}{x_1^-}, x_2\right) - \Psi\left(\frac{1}{x_1^+}, x_2\right)\\
&= \frac{1}{\mathrm{i}}\oint\frac{\mathrm{d}w}{2\pi i}\log(w - x_2)\left(w - \frac{1}{w}\right)\left[\frac{1}{\left(w - x_1^-\right)\left(w - \frac{1}{x_1^-}\right)} + \frac{1}{\left(w - x_1^+\right)\left(w - \frac{1}{x_1^+}\right)}\right]\\
&= -\frac{2}{\mathrm{i}}\log x_2 + \frac{1}{\mathrm{i}}\log\left(\frac{1}{x_1^-} - x_2\right)\left(\frac{1}{x_1^+} - x_2\right) = \frac{1}{\mathrm{i}}\log\left(1 - \frac{1}{x_1^- x_2}\right)\left(1 - \frac{1}{x_1^+ x_2}\right).
\end{aligned} \tag{A.4}$$

Analogously to the derivation above, we establish the following identities.

I For $|x_1^+| < 1\,,\ |x_1^-| > 1$ and $|x_2| > 1$ one has

$$\Psi\left(\frac{1}{x_1^-}, x_2\right) - \Psi\left(\frac{1}{x_1^+}, x_2\right) = -\frac{2}{\mathrm{i}}\log x_2 + \frac{1}{\mathrm{i}}\log\left(\frac{1}{x_1^-} - x_2\right)\left(x_1^+ - x_2\right). \tag{A.5}$$

[11] Perhaps, one can use the terminology ‘upper mirror’ and ‘down mirror’ to distinguish the regions obtained from the particle region by shifting the corresponding z-variables upward or downward the real axis, respectively.

II For $|x_1^+| > 1\,,\ |x_1^-| < 1$ and $|x_2| > 1$ one has

$$\Psi\left(\frac{1}{x_1^-}, x_2\right) - \Psi\left(\frac{1}{x_1^+}, x_2\right) = -\frac{2}{\mathrm{i}}\log x_2 + \frac{1}{\mathrm{i}}\log\left(x_1^- - x_2\right)\left(\frac{1}{x_1^+} - x_2\right). \tag{A.6}$$

III For $|x_1^+| > 1\,,\ |x_1^-| > 1$ and $|x_2| < 1$ one has

$$\Psi\left(\frac{1}{x_1^-}, x_2\right) - \Psi\left(\frac{1}{x_1^+}, x_2\right) = -\Psi\left(\frac{1}{x_1^-}, \frac{1}{x_2}\right) + \Psi\left(\frac{1}{x_1^+}, \frac{1}{x_2}\right) + \Psi\left(\frac{1}{x_1^-}, 0\right) - \Psi\left(\frac{1}{x_1^+}, 0\right)$$
$$= -\frac{1}{\mathrm{i}}\log\left(x_1^- - x_2\right)\left(x_1^+ - x_2\right) + \mathrm{i}\log\frac{g^2}{4}. \tag{A.7}$$

IV For $|x_1^+| < 1\,,\ |x_1^-| > 1$ and $|x_2| < 1$ one has

$$\Psi\left(\frac{1}{x_1^-}, x_2\right) - \Psi\left(\frac{1}{x_1^+}, x_2\right) = -\Psi\left(\frac{1}{x_1^-}, \frac{1}{x_2}\right) + \Psi\left(\frac{1}{x_1^+}, \frac{1}{x_2}\right) + \Psi\left(\frac{1}{x_1^-}, 0\right) - \Psi\left(\frac{1}{x_1^+}, 0\right)$$
$$= -\frac{2}{\mathrm{i}}\log x_2 - \frac{1}{\mathrm{i}}\log\left(\frac{1}{x_1^-} - \frac{1}{x_2}\right)\left(x_1^+ - \frac{1}{x_2}\right) + \frac{1}{\mathrm{i}}\log\frac{x_1^+}{x_1^-} + \mathrm{i}\log\frac{g^2}{4}. \tag{A.8}$$

In deriving this formula we made use of the identity (A.5).

V In proving the fourth relation we have used the fact that for $|x_1^+| > 1$, $|x_1^-| > 1$ and $|x_2| < 1$ the following identity holds:

$$\Psi\left(\frac{1}{x_1^-}, 0\right) - \Psi\left(\frac{1}{x_1^+}, 0\right) = \mathrm{i}\log\left(x_1^- x_1^+\right) + \mathrm{i}\log\frac{g^2}{4}\,, \tag{A.9}$$

while in proving the fifth one for $|x_1^+| < 1\,,\ |x_1^-| > 1$ and $|x_2| < 1$ we relied on

$$\Psi\left(\frac{1}{x_1^-}, 0\right) - \Psi\left(\frac{1}{x_1^+}, 0\right) = \frac{1}{\mathrm{i}}\log\frac{x_1^+}{x_1^-} + \mathrm{i}\log\frac{g^2}{4}\,. \tag{A.10}$$

Similar formulas exist for other values of $x_1^\pm$.

Appendix A.3. An alternative derivation of the dressing factor

In this appendix we provide an alternative derivation of the dressing factor for mirror bound states by picking up a bound state solution with all constituent particles being in the mirror region $\mathrm{Im}(x_j^\pm) < 0$, where

$$\mathrm{Im}(x_j^\pm) < 0\,, \quad j = 1, \ldots, Q\,, \quad |x_1^+| < 1\,, \quad |x_Q^-| > 1\,. \tag{A.11}$$

Most compactly, the corresponding solution can be written in terms of the function

$$\tilde{x}(u) = \frac{1}{2}\left(u - \mathrm{i}\sqrt{4 - u^2}\right)\,, \quad \mathrm{Im}\big(\tilde{x}(u)\big) < 0\,, \tag{A.12}$$

as follows:

$$x_j^- = \tilde{x}\left(u + \frac{\mathrm{i}}{g}(Q - 2j)\right)\,, \quad x_j^+ = \tilde{x}\left(u + \frac{\mathrm{i}}{g}(Q - 2j + 2)\right)\,, \quad j = 1, \ldots, Q\,. \tag{A.13}$$

The function $\tilde{x}(u)$ satisfies the following inequalities:

$$|\tilde{x}(u - \mathrm{i}y)| > 1\,, \quad |\tilde{x}(u + \mathrm{i}y)| < 1\,, \quad \text{for } u \in \mathbf{R} \text{ and } y > 0\,. \tag{A.14}$$

Thus, we find for odd Q case, $Q = 2m + 1$,

$$\begin{aligned} &\left|x_j^\pm\right| < 1 \ \text{ if } j = 1, \ldots, m \ \Rightarrow \ \text{anti-particle region}, \\ &\left|x_{m+1}^-\right| > 1,\ |x_{m+1}^+| < 1 \ \Rightarrow \ \text{mirror region}, \\ &\left|x_j^\pm\right| > 1 \ \text{ if } j = m+2, \ldots, Q \ \Rightarrow \ \text{particle region}. \end{aligned} \tag{A.15}$$

So, only one particle is in the mirror region.

If Q is even, $Q = 2m$, the story is more complicated and we find

$$
\begin{aligned}
&x_j^{\pm}| < 1 \;\text{ if } j = 1, \ldots, m-1 \;\Rightarrow\; \text{anti-particle region}\,, \\
&x_j^{\pm}| > 1 \;\text{ if } j = m+2, \ldots, Q \;\Rightarrow\; \text{particle region}\,, \\
&x_m^- = \tilde{x}(u)\,,\; |x_m^+| < 1 \;\Rightarrow \begin{cases} \text{anti-particle region if } x_m^- = \tilde{x}(u+\mathrm{i}0) \\ \text{mirror region if } x_m^- = \tilde{x}(u-\mathrm{i}0) \end{cases}, \\
&x_{m+1}^-| > 1\,,\; x_{m+1}^+ = \tilde{x}(u) \;\Rightarrow \begin{cases} \text{mirror region if } x_{m+1}^+ = \tilde{x}(u+\mathrm{i}0)\,, \\ \text{particle region if } x_{m+1}^+ = \tilde{x}(u-\mathrm{i}0) \end{cases}.
\end{aligned}
\tag{A.16}
$$

To apply the fusion procedure, as well as the formulas for the analytic continuation, it is convenient to rewrite the solution (A.13) in terms of the function $x(u)$ introduced in (3.9). This function satisfies the following inequalities:

$$
\mathrm{Im}\, x(u+\mathrm{i}y) > 0\,, \quad \mathrm{Im}\, x(u-\mathrm{i}y) < 0\,, \quad \text{for } u \in \mathbf{R} \text{ and } y > 0\,. \tag{A.17}
$$

The relation between $\tilde{x}(u)$ and $x(u)$ follows from equations (A.14) or (A.17):

$$
\tilde{x}(u-\mathrm{i}y) = x(u-\mathrm{i}y)\,, \quad \tilde{x}(u+\mathrm{i}y) = \frac{1}{x(u+\mathrm{i}y)}\,, \quad \text{for } u \in \mathbf{R} \text{ and } y > 0\,. \tag{A.18}
$$

Now we have two separate cases for Q odd and even.

- Thus, we find for odd Q case, $Q = 2m+1$,

$$
\begin{aligned}
&x_j^- = \frac{1}{x\left(u + \frac{\mathrm{i}}{g}(Q-2j)\right)}, \quad x_j^+ = \frac{1}{x\left(u + \frac{\mathrm{i}}{g}(Q-2j+2)\right)}, \quad j = 1, \ldots, m, \\
&x_{m+1}^- = x\left(u - \frac{\mathrm{i}}{g}\right), \quad x_{m+1}^+ = \frac{1}{x\left(u + \frac{\mathrm{i}}{g}\right)}, \\
&x_j^- = x\left(u + \frac{\mathrm{i}}{g}(Q-2j)\right), \quad x_j^+ = x\left(u + \frac{\mathrm{i}}{g}(Q-2j+2)\right), \quad j = m+2, \ldots, Q.
\end{aligned}
\tag{A.19}
$$

  Thus, only the middle particle is in the region $\mathcal{R}_1$.

- If Q is even, $Q = 2m$, the structure of the solution is more complicated

$$
\begin{aligned}
&x_j^- = \frac{1}{x\left(u + \frac{\mathrm{i}}{g}(Q-2j)\right)}, \quad x_j^+ = \frac{1}{x\left(u + \frac{\mathrm{i}}{g}(Q-2j+2)\right)}, \quad j = 1, \ldots, m-1\,, \\
&x_m^+ = \frac{1}{x\left(u + \frac{2\mathrm{i}}{g}\right)}, \qquad x_m^- = \begin{cases} \frac{1}{x(u+\mathrm{i}0)}, & z_m \in \text{anti-particle region} \\ x(u-\mathrm{i}0), & z_m \in \text{mirror region} \end{cases}, \\
&x_{m+1}^- = x\left(u - \frac{2\mathrm{i}}{g}\right), \qquad x_{m+1}^+ = \begin{cases} \frac{1}{x(u+\mathrm{i}0)}, & z_m \in \text{mirror region} \\ x(u-\mathrm{i}0), & z_m \in \text{particle region} \end{cases}, \\
&x_j^- = x\left(u + \frac{\mathrm{i}}{g}(Q-2j)\right), \quad x_j^+ = x\left(u + \frac{\mathrm{i}}{g}(Q-2j+2)\right), \quad j = m+2, \ldots, Q.
\end{aligned}
\tag{A.20}
$$

  Again, only one particle is in the region $\mathcal{R}_1$.

For simplicity, we consider the case of $Q = 2m+1$ and take the second particle in the dressing factor to be a fundamental one of string theory. Then, m particles from the bound state solution occur in the anti-particle region $\mathcal{R}_2$, m particles are in the particle region and one particle is in $\mathcal{R}_1$. By using our analytic continuation formulas for dressing phases of particles in the corresponding regions, we get

$$\theta(y_1, y_2) = \Phi\left(y_1^+, y_2^+\right) - \Psi\left(y_1^+, y_2^+\right) - \Phi\left(y_1^+, y_2^-\right) + \Psi\left(y_1^+, y_2^-\right) \tag{A.21}$$

$$-\Phi\left(y_1^-, y_2^+\right) + \Phi\left(y_1^-, y_2^-\right) + \frac{1}{\mathrm{i}}\log\prod_{j=1}^{m}\frac{\frac{1}{x_j^-} - y_2^+}{x_j^- - y_2^+}\frac{x_j^- - y_2^-}{\frac{1}{x_j^-} - y_2^-}.$$

Taking into account formula (6.5), for the factor Σ^{Q1} we find

$$\frac{1}{\mathrm{i}}\log\Sigma^{Q1}(y_1, y_2) = \Phi\left(y_1^+, y_2^+\right) - \Psi\left(y_1^+, y_2^+\right) - \Phi\left(y_1^+, y_2^-\right) + \Psi\left(y_1^+, y_2^-\right) \tag{A.22}$$

$$- \Phi\left(y_1^-, y_2^+\right) + \Phi\left(y_1^-, y_2^-\right)$$

$$+\frac{1}{\mathrm{i}}\log\frac{1 - \frac{1}{y_1^+ y_2^-}}{1 - \frac{1}{y_1^- y_2^+}}\prod_{j=1}^{m}\frac{y_2^+}{y_2^-}\frac{x_j^- - y_2^-}{x_j^- - y_2^+}\prod_{j=m+1}^{2m}\frac{1 - \frac{1}{x_j^- y_2^-}}{1 - \frac{1}{x_j^- y_2^+}}.$$

By using equations (A.19), this formula can be written in the following form:

$$\frac{1}{\mathrm{i}}\log\Sigma^{Q1}(y_1, y_2) = \Phi\left(y_1^+, y_2^+\right) - \Psi\left(y_1^+, y_2^+\right) - \Phi\left(y_1^+, y_2^-\right) + \Psi\left(y_1^+, y_2^-\right) \tag{A.23}$$

$$- \Phi\left(y_1^-, y_2^+\right) + \Phi\left(y_1^-, y_2^-\right) + \frac{1}{\mathrm{i}}\log\frac{1 - \frac{1}{y_1^+ y_2^-}}{1 - \frac{1}{y_1^- y_2^+}}\prod_{j=1}^{Q-1}\frac{1 - \frac{1}{x\left(u+\frac{\mathrm{i}}{g}(Q-2j)\right)y_2^-}}{1 - \frac{1}{x\left(u+\frac{\mathrm{i}}{g}(Q-2j)\right)y_2^+}},$$

which is obviously the same expression as (6.9). Considerations of the case $Q = 2m$ is analogous and leads to the same conclusion—two different bound state solutions produce one and the same dressing factor.

Appendix A.4. Details on the derivation of the mirror bound state dressing factor

In section 6 we constructed the dressing factor for bound states of the mirror theory. This construction requires the use of further identities for the Ψ-function which we present here. Recall that the parameters $y_1^+ = x_1^+$, $y_1^+ = x_Q^-$ and $y_2^+ = x_2^+$, $y_2^+ = x_Q'^-$ of the mirror bound states (6.1) satisfy the relations

$$y_1^+ + \frac{1}{y_1^+} - y_1^- - \frac{1}{y_1^-} = \frac{2\mathrm{i}}{g}Q, \qquad y_2^+ + \frac{1}{y_2^+} - y_2^- - \frac{1}{y_2^-} = \frac{2\mathrm{i}}{g}Q'.$$

Consider now a particular solution (6.7) of the bound state equations (6.1). If both bound states involved in the construction of the dressing factor are of this type, then

$$|y_1^+| < 1\,, \quad |y_1^-| > 1\,, \quad |y_2^+| < 1 \quad |y_2^-| > 1\,. \tag{A.24}$$

By using equation (5.13), for this kinematic configuration, we find

$$\Psi\left(y_1^+, y_2^-\right) - \Psi\left(y_1^-, y_2^-\right) = \frac{2Q}{\mathrm{i}}\log y_2^- - \frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q}\left(w_j^+\left(y_1^-\right) - y_2^-\right)\left(w_j^-\left(y_1^+\right) - y_2^-\right). \tag{A.25}$$

Taking into account that for solution (6.7) equations (5.3) acquire the form

$$w_Q^+(y_1^-) = y_1^+ = x_1^+ = \frac{1}{x\left(u + \frac{\mathrm{i}}{g}Q\right)}\,, \qquad w_j^-(y_1^+) = \frac{1}{x_j^-}\,,$$

$$w_Q^-(y_1^+) = \frac{1}{y_1^-} = \frac{1}{x_Q^-} = \frac{1}{x\left(u - \frac{\mathrm{i}}{g}Q\right)}\,, \tag{A.26}$$

where x_j^- is given by equation (6.7), we obtain the following identity:

$$\Psi\left(y_1^+,y_2^-\right)-\Psi\left(y_1^-,y_2^-\right)=-\frac{1}{\mathrm{i}}\log\left(\frac{y_1^+}{y_2^-}-1\right)\left(\frac{1}{y_1^-y_2^-}-1\right)\prod_{j=1}^{Q-1}\left(1-\frac{1}{x_j^-y_2^-}\right)^2. \quad \text{(A.27)}$$

The latter formula also implies the following relation:

$$\Psi\left(y_2^+,y_1^-\right)-\Psi\left(y_2^-,y_1^-\right)=-\frac{1}{\mathrm{i}}\log\left(\frac{y_2^+}{y_1^-}-1\right)\left(\frac{1}{y_1^-y_2^-}-1\right)\prod_{k=1}^{Q'-1}\left(1-\frac{1}{y_1^-z_k^-}\right)^2, \quad \text{(A.28)}$$

where $z_k^-=1/w_k^-(y_2^+)=x(u_2+\frac{\mathrm{i}}{g}(Q'-2k))$. Further, by applying the basic identity (3.15) to equations (A.27) and (A.28), we find the other two relations

$$\Psi(y_1^+,y_2^+)-\Psi(y_1^-,y_2^+)=\frac{Q}{\mathrm{i}}\log\frac{g^2}{4}-\frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q-1}\left(1-\frac{1}{x_j^-y_2^+}\right)^2 \quad \text{(A.29)}$$
$$+\frac{1}{\mathrm{i}}\log\left(y_2^+-y_1^-\right)\left(y_2^+-\frac{1}{y_1^+}\right)\prod_{j=1}^{Q-1}\left(u_1-u_2+\frac{\mathrm{i}}{g}(Q-Q'-2j)\right)^2,$$

$$\Psi\left(y_2^+,y_1^+\right)-\Psi\left(y_2^-,y_1^+\right)=\frac{Q'}{\mathrm{i}}\log\frac{g^2}{4}-\frac{1}{\mathrm{i}}\log\prod_{k=1}^{Q'-1}\left(1-\frac{1}{y_1^+z_k^-}\right)^2 \quad \text{(A.30)}$$
$$+\frac{1}{\mathrm{i}}\log\left(y_1^+-y_2^-\right)\left(y_1^+-\frac{1}{y_2^+}\right)\prod_{k=1}^{Q'-1}\left(u_2-u_1+\frac{\mathrm{i}}{g}(Q'-Q-2k)\right)^2$$

valid for kinematic parameters $y_{1,2}^{\pm}$ obeying inequalities (A.24). In deriving these formulas we have used the fact that

$$\Psi\left(\frac{1}{y_1^+},0\right)-\Psi\left(\frac{1}{y_1^-},0\right)=\frac{Q}{\mathrm{i}}\log\frac{g^2}{2}-\frac{1}{\mathrm{i}}\log\left[w_Q^+(y_1^-)w_Q^-(y_1^+)\prod_{j=1}^{Q-1}\left(w_j^-(y_1^+)\right)\right]$$
$$=\frac{Q}{\mathrm{i}}\log\frac{g^2}{2}-\frac{1}{\mathrm{i}}\log\frac{y_1^+}{y_1^-}\prod_{j=1}^{Q-1}\frac{1}{(x_j^-)^2} \quad \text{(A.31)}$$

and also the identity

$$\left(x_j^--y_2^-\right)\left(1-\frac{1}{x_j^-y_2^-}\right)=u_1-u_2+\frac{\mathrm{i}}{g}(Q-Q'-2j), \quad \text{(A.32)}$$

where u_1 and u_2 are the corresponding rapidity parameters

$$u_1=x_1^++\frac{1}{x_1^+}-Q\frac{\mathrm{i}}{g}=x_1^-+\frac{1}{x_1^-}+Q\frac{\mathrm{i}}{g},\qquad u_2=x_2^++\frac{1}{x_2^+}-Q'\frac{\mathrm{i}}{g}=x_2^-+\frac{1}{x_2^-}+Q'\frac{\mathrm{i}}{g}.$$

Finally, using formulas (A.27)–(A.30), we find

$$-\Psi\left(y_1^+,y_2^+\right)+\Psi\left(y_1^+,y_2^-\right)+\Psi\left(y_2^+,y_1^+\right)-\Psi\left(y_2^+,y_1^-\right)+\frac{1}{\mathrm{i}}\log\prod_{j=1}^{Q-1}\frac{1-\frac{1}{x_j^-y_2^-}}{1-\frac{1}{x_j^-y_2^+}}\prod_{k=1}^{Q'-1}\frac{1-\frac{1}{y_1^+z_k^-}}{1-\frac{1}{y_1^-z_k^-}}$$
$$=\frac{1}{2}\left(-\Psi\left(y_1^+,y_2^+\right)+\Psi\left(y_1^+,y_2^-\right)+\Psi\left(y_2^+,y_1^+\right)-\Psi\left(y_2^+,y_1^-\right)\right)$$
$$+\frac{1}{2}\left(-\Psi\left(y_1^-,y_2^+\right)+\Psi\left(y_1^-,y_2^-\right)+\Psi\left(y_2^-,y_1^+\right)-\Psi\left(y_2^-,y_1^-\right)\right) \quad \text{(A.33)}$$

$$-\frac{1}{\mathrm{i}}\log\frac{\prod_{j=1}^{Q-1}\frac{g}{2}\left(u_1-u_2+\frac{\mathrm{i}}{g}(Q-Q'-2j)\right)}{\prod_{k=1}^{Q'-1}\frac{g}{2}\left(u_2-u_1+\frac{i}{g}(Q'-Q-2k)\right)}+\frac{1}{2\mathrm{i}}\log\frac{y_1^+y_2^-}{y_1^-y_2^+}.$$

One more formula we need to complete the derivation is

$$\frac{\prod_{k=1}^{Q'-1}\frac{g}{2}\left(u_2-u_1+\frac{\mathrm{i}}{g}(Q'-Q-2k)\right)}{\prod_{j=1}^{Q-1}\frac{g}{2}\left(u_1-u_2+\frac{\mathrm{i}}{g}(Q-Q'-2j)\right)}\frac{\Gamma\left[1-\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}{\Gamma\left[1+\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}$$
$$=\mathrm{i}^{Q-Q'}\frac{\Gamma\left[Q'-\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}{\Gamma\left[Q+\frac{\mathrm{i}}{2}g\left(y_1^++\frac{1}{y_1^+}-y_2^+-\frac{1}{y_2^+}\right)\right]}. \quad \text{(A.34)}$$

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